Kripke semantics augmented with derivability

Mojtaba Mojtahedi (Ghent University)

January 17, 2023

GL and its Classical models

- $\mathsf{GL} := \mathsf{K} + \Box(\Box A \to A) \to \Box A$
- $\mathcal{K} := (W, \sqsubset, \models)$:
  - $(W, \sqsubset)$ transitive and conversely well-founded
  - $\mathcal{K}, w \models \Box A$ iff for all $u \sqsupset w$ we have $\mathcal{K}, u \models A$.
- GL is sound and complete for finite Kripke models.
- A well-known benefit of fmp: Solovey's proof of arithmetical completeness of GL for provability interpretations.

Provability semantics

$$\mathcal{K} = (W, \sqsubset, \models, \{\Gamma_w\}_{w \in W})$$

- $(W, \sqsubset)$ is transitive and conversely well-founded
- $A \in \Gamma_w$ implies $\mathcal{K}, w \models A$
- $\mathcal{K}, w \models \Box A$ implies $\Box A \in \Gamma_w$

$$\mathcal{K}, w \models \Box A \quad \Leftrightarrow \quad \forall u \sqsupset w (\Gamma_u \vdash_{\mathsf{GL}} A)$$

Provability semantics

$$\mathcal{K} = (W, \sqsubset, \models, \{\Gamma_w\}_{w \in W})$$

- $(W, \sqsubset)$ is transitive and conversely well-founded
- $A \in \Gamma_w$ implies $\mathcal{K}, w \models A$
- $\mathcal{K}, w \models \Box A$ implies $\Box A \in \Gamma_w$

$$\mathcal{K}, w \models \Box A \quad \Leftrightarrow \quad \forall u \sqsupset w (\Gamma_u \vdash_{\mathsf{GL}} A)$$

Circular definition?

Soundness

Theorem

Let $\mathcal{K} = (W, \sqsubset, \models, \{\Gamma_w\}_{w \in W})$ *be a provability semantic. Then* $\mathcal{K} \models \mathsf{GL}$.

Proof.

We use induction on the proof $\mathsf{GL} \vdash A$ and show $\mathcal{K}, w \models A$.

- $\mathcal{K}, w \models \Box(\Box A \to A) \to \Box A$. Let $\mathcal{K}, w \models \Box(\Box A \to A)$. Hence for every $u \sqsupset w$ we have $\Gamma_u \vdash_{\mathsf{GL}} \Box A \to A$. By induction on $u \sqsupset w$ we may show $\mathcal{K}, u \models \Box A$ and hence $\Gamma_u \vdash_{\mathsf{GL}} A$.
- Necessitation. Let $\mathsf{GL} \vdash \Box A$ derived by $\mathsf{GL} \vdash A$. Hence for every $u \sqsupset w$ we have $\Gamma_u \vdash_{\mathsf{GL}} A$ and thus $\mathcal{K}, w \models \Box A$. ❑

Example

Every Kripke model $\mathcal{K}_0 = (W, \sqsubset, \models)$ can be considered as a provability semantic.
$\mathcal{K} = (W, \sqsubset, \models, \{\Gamma_w\}_{w \in W})$ with

$$\Gamma_w := \{A : \mathcal{K}_0, w \models A\}.$$

Using induction on $w \in W$ one may show

$$\mathcal{K}_0, w \models A \quad \text{iff} \quad \mathcal{K}, w \models A$$

Completeness

As a consequence of the previous example:

Theorem

GL *is complete for provability semantics.*

What is extra benefit of provability semantics?

Complicated axiom-schemas show up:

- $\Box\neg\neg\Box A \to \Box\Box A$. A generalization of these axioms, are called Visser axiom schemas.
- $\Box(A \vee B) \to \Box(\Box A \vee B)$. Leivant axiom.

Frame property for such weird axioms?

Frame property for such weird axioms?

Rosalie Iemhoff proves soundness-completess for some Kripke semantics.

Regrettably, such Kripke models are infinite.

Provability semantics for intuitionistic provability logics

- We defined provability semantics for intuitionistic provability logics.
- We showed the finite model property and decidability for the provability logic of HA.
- Via such finite provability semantics, we were able to prove the arithmetical completeness result for the provability logic of HA.

[1] Mojtahedi, Mojtaba. “On Provability Logic of HA.” arXiv preprint arXiv:2206.00445 (2022).

Inruitionistic provability semantics

- As one expects, the intuitionistic provability semantics, has an extra relation $\preccurlyeq$ for the intuitionstic $\to$.

$$\mathcal{K} = (W, \preccurlyeq, \sqsubset, \Vdash, \{\Gamma_w\}_{w \in W})$$

Inruitionistic provability semantics II

- We restrict Γ_w in the definition, for technical reasons.
- Given two sets Δ and Γ of proposuitions and $\varphi_w \in \Gamma$ such that Δ and Γ are closed under Δ-conjunctions ($B \in \Delta$ and $C \in \Gamma$ implies $B \wedge C \in \Gamma$), we assume

$$\Gamma_w := \overbrace{\{A \in \Delta : \mathcal{K}, w \models A\}}^{\Delta_w} \cup \{\varphi_w\}$$

- Thus Γ_w includes all locally true propositions in Δ together with a single proposition $\varphi_w \in \Gamma$ which might not be in Δ.
- Also we consider the general case T instead of GL:

$$\mathcal{K}, w \Vdash \Box A \quad \text{iff} \quad \forall\, u \sqsupset w \; (\Gamma_w \vdash_{\mathsf{T}} A)$$

Inruitionistic provability semantics III

Definition

Such models are called $(\Delta, \Gamma, \mathsf{T})$-semantics, and annotated as

$$\mathcal{K} = (W, \preccurlyeq, \sqsubset, \Vdash, \{\varphi_w\}_{w \in W})$$

Whenever $\Gamma = \Delta$ we simply say that $\mathcal{K}$ is a (Γ, T)-semantic. In this case it doesn't matter how $\varphi_w \in \Gamma$ are defined.

The proof of following theorem is straightforward:

Theorem

The Σ_1*-provability logic of* HA *is sound and complete for* $(\mathsf{SNNIL}, \mathsf{iGLC_a})$*-models.*

Soundness and completeness results

Neverthless, the following theorem is not trivial:

Theorem

The provability logic of HA *is sound and complete for* $(\mathsf{SNNIL}(\Box), \mathsf{C}{\downarrow}\mathsf{SN}(\Box), \mathsf{iGL})$*-models.*

One may use the previous two results to reduce arithmetical completeness to the one for Σ_1-substitutions.

Preservativity

- $A \mathrel{\substack{\mathsf{T} \\ \approx \\ \Gamma}} B$ iff $\forall E \in \Gamma(\mathsf{T} \vdash E \to A \Rightarrow \mathsf{T} \vdash E \to B)$.

Relation to preservativity

- Intuitionistic provability, is closely related to admissibility and also preservativity.
- Rosalie Iemhoff and Albert Visser showed such tight interactions between them.
- In the context of preservativity, weird axioms of the intuitionistic provability, gets more elegant form.
- Rosalie Iemhoff proves the completeness of several preservativity logics for Kripke models. Again the Kripke models are mainly infinite.
- Our provability semantics, can be extended to preservativity as well.

Extending to preservativity

For a $(\Delta, \Gamma, \mathsf{T})$-semantic $\mathcal{K}$, we extend $\mathcal{K}, w \Vdash A$ to the language with binary modal operator $\rhd$:

$$\begin{aligned} &\mathcal{K}, w \Vdash B \rhd C \quad \Leftrightarrow \\ &\forall u \sqsupset w \ \forall E \in \Delta \ (\Delta_u, \varphi_u \vdash_{\mathsf{T}} E \to B \text{ implies } \Delta_u, \varphi_u \vdash_{\mathsf{T}} E \to C), \end{aligned}$$

Note that in the above definition, B and C are considered in usual modal language. An extension to the full language of preservatitivity is still missing.

Theorem

$\overset{\mathsf{T}}{\underset{\Gamma}{|\!\approx}}$ is sound for $(\Delta, \Gamma, \mathsf{T})$-semantics, i.e. given such preservativity semantics $\mathcal{K}$, we have $\mathcal{K} \Vdash A \rhd B$ whenever $A \overset{\mathsf{T}}{\underset{\Gamma}{|\!\approx}} B$.

Proof.

Let $A \overset{\mathsf{T}}{\underset{\Gamma}{|\!\approx}} B$ and $\mathcal{K} = (W, \preccurlyeq, \sqsubset, V, \{\varphi_w\}_{w \in W})$ be a $(\Delta, \Gamma, \mathsf{T})$-semantics and $w \sqsubset u \in W$ and $E \in \Delta$ such that $\varphi_u, \Delta_u, E \vdash_{\mathsf{T}} A$. Hence there is a finite set $\Phi_u \subseteq \Delta_u$ such that $\Phi_u, E, \varphi_u \vdash A$. By conjunctive closure condition, we have $\bigwedge \Phi_u \wedge E \wedge \varphi_u \in \Gamma$ and thus by $A \overset{\mathsf{T}}{\underset{\Gamma}{|\!\approx}} B$ we get $\Phi_u, E, \varphi_u \vdash_{\mathsf{T}} B$. Hence we have $\varphi_u, \Delta_u, E \vdash_{\mathsf{T}} B$. ❑

Greatest lower bound (glb)

- B is a (Γ, T)-lb for A if:
  1. $B \in \Gamma$,
  2. $\mathsf{T} \vdash B \to A$.
- B is the (Γ, T)-glb for A, if for every (Γ, T)-lb B' for A we have $\mathsf{T} \vdash B' \to B$.
- Up to T-provable equivalence relation, such glb is unique and we annotate it as $\lfloor A \rfloor_{\Gamma}^{\mathsf{T}}$.
- (Γ, T) is downward compact, if every $A \in \mathcal{L}_\Box$ has a (Γ, T)-glb $\lfloor A \rfloor_{\Gamma}^{\mathsf{T}}$.
- If $\lfloor A \rfloor_{\Gamma}^{\mathsf{T}}$ can be effectively computed, we say that (Γ, T) is recursively downward compact.

Theorem

$(\mathsf{NNIL}, \mathsf{IPC})$ *is recursively downward compact.*

(Γ, T)-glb and $\underset{\Gamma}{\overset{\mathsf{T}}{\mid\!\approx}}$

Theorem

B is the (Γ, T)-glb for A iff

- $B \in \Gamma$,
- $\mathsf{T} \vdash B \to A$,
- $A \underset{\Gamma}{\overset{\mathsf{T}}{\mid\!\approx}} B$.

Hence we have $A \underset{\Gamma}{\overset{\mathsf{T}}{\mid\!\approx}} \lfloor A \rfloor_{\Gamma}^{\mathsf{T}}$.

Corollary

If $\lfloor A \rfloor_{\Gamma}^{\mathsf{T}}$ exists, then for every $B \in \mathcal{L}_{\Box}$ we have

$$\mathsf{T} \vdash \lfloor A \rfloor_{\Gamma}^{\mathsf{T}} \to B \quad \textit{iff} \quad A \underset{\Gamma}{\overset{\mathsf{T}}{\mid\!\approx}} B.$$

Theorem

Forcing relationship for finite $(\Delta, \Gamma, \mathsf{T})$-semantic is decidable whenever (Δ, T) is recursively downward compact and T is sound.

Proof.

Let $\mathcal{K} = (W, \preccurlyeq, \sqsubset, V, \varphi)$ be a $(\Delta, \Gamma, \mathsf{T})$-semantic. We show decidability of $\mathcal{K}, w \Vdash A$ by double induction on W ordered by $\sqsupset$ and complexity of A.

- $A = \Box B$. It is enough to decide $\Delta_u \vdash_{\mathsf{T}} \varphi_u \to B$ for every $u \sqsupset w$. Since (Δ, T) is recursively downward compact, one may effectively compute $\lfloor \varphi_u \to B \rfloor_{\Delta}^{\mathsf{T}}$. By definition of $\lfloor . \rfloor_{\Gamma}^{\mathsf{T}}$ it is enough to decide $\Delta_u \vdash_{\mathsf{T}} \lfloor \varphi_u \to B \rfloor_{\Delta}^{\mathsf{T}}$ which is equivalent to $\mathcal{K}, u \Vdash \lfloor \varphi_u \to B \rfloor_{\Delta}^{\mathsf{T}}$. Then use induction hypothesis. ❑

Future works (I had some failed attempts)

Question

A finite provability semantics for the Iemhoffs prservativity logic iPH *is desired.*

Answering above question is important because it may casue a solution to a conjecture posed by Iemhoff for arithmetical completeness of iPH.

Future works (I didn't tried!)

- Interpretability, is tightly related to preservativity. Currently there is some Kripke-style sematic for the interpretability, invented by Veltman. Is it possible to adapt provability semantics for interpretability?
- Use provability semantics for the study of admissibility and preservativity in classical GL.

Thanks For Your Attention