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# On Provability Logic of HA

Mojtaba Mojtahedi (University of Tehran)

University of Tehran

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# Propositional non-modal language

$$\mathcal{L}_0: \quad \vee, \wedge, \rightarrow, \mathsf{par}, \mathsf{var}$$

- $\mathsf{var}$ and $\mathsf{par}$ are countably infinite sets of atomics and $\top, \bot \in \mathsf{par}$.

- $\neg A := A \rightarrow \bot$.

- $\mathsf{atom} := \mathsf{par} \cup \mathsf{var}$

- $\mathsf{par}$ stands for $\Sigma_1$-substitutions, $\mathsf{var}$ for arbitrary.

- For a propositional substitution $\theta$, by default $\theta(p) := p$ for every $p \in \mathsf{par}$.

- $\mathcal{L}_0(X)$ indicates the set of all Boolean combinations of propositions in the set $X$.

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## Modal language

$$\mathcal{L}_\rhd := \mathcal{L}_0 + \rhd \quad \text{and} \quad \Box A := \top \rhd A \quad \text{and} \quad \mathcal{L}_\Box := \mathcal{L}_0 + \Box$$

- $\rhd$ is a binary modal operator.
- We usually consider $A \rhd B$ for preservativity.
- $\mathsf{B} := \{\Box A : A \in \mathcal{L}_\Box\}$.
- $\mathsf{parb} := \mathsf{par} \cup \mathsf{B}$.
- $\mathsf{atomb} := \mathsf{atom} \cup \mathsf{B}$.
- $\boxdot A := A \wedge \Box A$.

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## Logics

K: $\square(A \to B) \to (\square A \to \square B)$.

4: $\square A \to \square\square A$.

L: $\square(\square A \to A) \to \square A$. (The Löb's axiom)

$C_p$: $p \to \square p$ for every $p \in$ par.

$C_a$: $a \to \square a$ for every $a \in$ atom.

Given a logic L and axiom-schemata $X_1, \ldots, X_n$, the logic $LX_1 \ldots X_n$ is defined as L plus the axioms $X_1, \ldots, X_n$. Then we define following modal logics:

- i: IPC plus necessitation and $C_p$.
- iGL := iK4L.

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## Propositional substitutions

- $\theta(x)$ is a proposition in the language $\mathcal{L}_\triangleright$ for every $x \in \mathsf{var}$.
- $\theta(p) = p$ for every $p \in \mathsf{par}$.
- $\theta(B \circ C) = \theta(B) \circ \theta(C)$ for every $\circ \in \{\vee, \wedge, \rightarrow, \triangleright\}$.

Given $\theta$, define $\hat{\theta}$ same as $\theta$ except for boxed propositions for which $\hat{\theta}$ operates as identity:

$$\hat{\theta}(A \triangleright B) := A \triangleright B \text{ and hence } \hat{\theta}(\square A) := \square A.$$

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## Heyting Arithmetic

The Heyting arithmetic is defined as the intuitionistic fragment of first-oder Peano Arithmetic PA.

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## Arithmetical substitutions

<span>◄ iPH</span>

A function $\alpha$ on atom such that $\alpha(a)$ is a first-order arithmetical sentence for every $a \in$ atom and $\alpha(a) \in \Sigma_1$ for every $a \in$ par and $\alpha(\bot) = \bot$ and $\alpha(\top) = \top$. Moreover $\alpha$ is called a $\Sigma_1$-substitution if $\alpha(a) \in \Sigma_1$ for every $a \in$ atom.

- $\alpha_{\mathsf{HA}}(a) := \alpha(a)$ for every $a \in$ atom, and $\alpha_{\mathsf{HA}}(\bot) = \bot$.
- $\alpha_{\mathsf{HA}}$ commutes with boolean connectives: $\vee, \wedge$ and $\rightarrow$.
- $\alpha_{\mathsf{HA}}(A \triangleright B)$ is defined as an arithmetization of $\Sigma_1$-preservativity: For every $E \in \Sigma_1$,

$$\text{if } \mathsf{HA} \vdash E \rightarrow \alpha_{\mathsf{HA}}(A) \text{ then } \mathsf{HA} \vdash E \rightarrow \alpha_{\mathsf{HA}}(B).$$

- $\alpha_{\mathsf{HA}}(\square A) =$ an arithmetization of "$A$ is provable in $\mathsf{HA}$".

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$\mathsf{PL}^\Box(\mathsf{HA})$, the provability logic of $\mathsf{HA}$ is defined as

$\{A \in \mathcal{L}_\Box : \mathsf{HA} \vdash \alpha_{\mathsf{HA}}(A) \text{ for every arithmetical substitution } \alpha\}$

$\mathsf{PL}^\triangleright(\mathsf{HA})$, the Preservativity logic of $\mathsf{HA}$ is defined

$\{A \in \mathcal{L}_\triangleright : \mathsf{HA} \vdash \alpha_{\mathsf{HA}}(A) \text{ for every arithmetical substitution } \alpha\}$

Similarly one may define $\mathsf{PL}^\Box_\Sigma(\mathsf{HA})$ and $\mathsf{PL}^\triangleright_\Sigma(\mathsf{HA})$ as provability and preservativity logics for $\Sigma_1$-substitutions.

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- $\mathsf{PL}^\Box(\mathsf{HA}) \nvdash \Box(A \vee B) \to (\Box A \vee \Box B)$, Myhill [1973]; Friedman [1975]

- $\mathsf{PL}^\Box(\mathsf{HA}) \vdash \Box(A \vee B) \to \Box(\boxdot A \vee \boxdot B)$, in which $\boxdot A$ is a shorthand for $A \wedge \Box A$, Leivant [1975]

- $\mathsf{PL}^\Box(\mathsf{HA}) \vdash \Box\neg\neg \Box A \to \Box\Box A$ and
  $\mathsf{PL}^\Box(\mathsf{HA}) \vdash \Box(\neg\neg \Box A \to \Box A) \to \Box(\Box A \vee \neg \Box A)$, Visser [1981, 1982]

- Decidability of letterless fragment of $\mathsf{PL}^\Box(\mathsf{HA})$. Visser [2002]

- Axiomatization and decidability of $\mathsf{PL}^\Box_\Sigma(\mathsf{HA})$. Ardeshir and Mojtahedi [2018]; Visser and Zoethout [2019]

- Axiomatization and decidability of $\mathsf{PL}^\Box_\Sigma(\mathsf{HA})$ relative in $\mathsf{PA}$ and $\mathbb{N}$. Mojtahedi [2021]

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The following translation, is some variant of the Gödel's celebrated translation for the embedding of IPC in S4 [Gödel, 1933].

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### Definition

For every proposition $A \in \mathcal{L}_\square$ define $A^\square$ inductively as follows:

- $A^\square := \boxdot A$, for $A \in \mathsf{var}$.

- $A^\square := A$ for $A \in \mathsf{parb}$.

- $(B \circ C)^\square := B^\square \circ C^\square$. for $\circ \in \{\vee, \wedge\}$.

- $(B \to C)^\square := \boxdot (B^\square \to C^\square)$.

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- $A \in \mathcal{L}_\square$ is called self complete if there is some $B \in \mathcal{L}_\square$ such that $A = B^\square$:

$$\mathsf{S} := \{B^\square : B \in \mathcal{L}_\square\}.$$

- $A$ is called $\mathsf{T}$-complete if $\mathsf{T} \vdash A \to \square A$:

$$\mathsf{C}^{\mathsf{T}} := \{A \in \mathcal{L}_\square : \mathsf{T} \vdash A \to \square A\}.$$

- If $\mathsf{T} \supseteq \mathsf{iK4}$ we have $\mathsf{S} \subseteq \mathsf{C}^{\mathsf{T}}$.

- We may omit the superscript $\mathsf{T}$ in the notation $\mathsf{C}^{\mathsf{T}}$ and simply write $\mathsf{C}$.

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A Kripke model for the intuitionistic modal logic, is a combination of a Kripke model for intuitionistic logic and the classical modal logic. Let $\mathcal{K} = (W, \prec, \sqsubset, V)$:

- $W \neq \emptyset$.

- $(W, \prec)$ is a partial order (transitive and irreflexive). We write $\preccurlyeq$ for the reflexive closure of $\prec$.

- $V$ is the valuation on atomics, i.e. $V \subseteq W \times \mathsf{atom}$.

- $w \preccurlyeq u$ and $w \, V \, a$ implies $u \, V \, a$ for every $w, u \in W$ and $a \in \mathsf{atom}$.

- $(\preccurlyeq; \sqsubset) \subseteq \sqsubset$, i.e. $w \preccurlyeq u \sqsubset v$ implies $w \sqsubset v$. This condition is assumed to ensure that the previous property holds for all modal propositions and not only for $a \in \mathsf{atom}$.

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$V$ is extended to all modal propositions:

- $\mathcal{K}, w \Vdash a$ iff $w \, V \, a$, for $a \in$ atom.

- $\mathcal{K}, w \Vdash A \wedge B$ iff $\mathcal{K}, w \Vdash A$ and $\mathcal{K}, w \Vdash B$.

- $\mathcal{K}, w \Vdash A \vee B$ iff $\mathcal{K}, w \Vdash A$ or $\mathcal{K}, w \Vdash B$.

- $\mathcal{K}, w \Vdash A \to B$ iff for every $u \succcurlyeq w$ if we have $\mathcal{K}, w \Vdash A$ then $\mathcal{K}, w \Vdash B$.

- $\mathcal{K}, w \Vdash A \triangleright B$ iff for every $u \sqsupset w$ with $\mathcal{K}, u \Vdash A$ we have $\mathcal{K}, w \Vdash B$.

- $\mathcal{K}, w \Vdash \Box A$ iff for every $u \sqsupset w$ we have $\mathcal{K}, w \Vdash A$.

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- We say that $u$ is a successor of $w$ if $w \sqsubset u$.

- We say that $u$ is a predecessor of $w$ if $u \sqsubset w$.

- We say that $u$ is above $w$ if $w \preccurlyeq u$.

- We say that $u$ is beneath $w$ if $u \preccurlyeq w$.

- We say that $u$ is generated by $w$ if $w \, (\overline{\sqsubset \cup \preccurlyeq}) \, u$.

- $\overline{S}$ indicates the reflexive transitive closure of $S$.

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### Theorem

*iGL is sound and complete for good Kripke models. Also* iGLC$_a$ *is sound and complete for good* C$_a$ *Kripke models.*

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- The class of *No Nested Implications to the Left*, NNIL formulae, for the nonmodal language $\mathcal{L}_0$, was introduced in [Visser et al., 1995], and more explored in [Visser, 2002].

- Visser et al. [1995] chracterize the NNIL via Kripke semantics.

- $A \in$ NNIL and $A \in$ NI for every $A \in$ atomb.

- $B \circ C \in$ NNIL if $B, C \in$ NNIL. Also $B \circ C \in$ NI if $B, C \in$ NI. ($\circ \in \{\vee, \wedge\}$)

- $B \to C \in$ NNIL if $B \in$ NI and $C \in$ NNIL.

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- One of the most interesting features of the Gödel-Löb axiom, is the fixed-point theorem.

- It is the propositional remainder of the Gödels diagnalization lemma.

- It says that if $x$ only appears in the scope of $\Box$ in $A$, then there is some $D$ such that $\mathsf{GL} \vdash D \leftrightarrow A[x : D]$. [Smoryński, 1985]

- One may generalize the same fixed-point theorem to iGL. [Iemhoff et al., 2005]

- It is well-known that one may generalize this fixed-point theorem to a simultaneous version.

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◀ 1st-red

### Theorem

*Let $\overrightarrow{E} := \{E_1, \ldots, E_m\}$ and $\overrightarrow{a} = \{a_1, \ldots, a_m\}$ such that every occurrences of $a_i$ in $E_j$ is in the scope of some $\square$. Then there is a substitution $\tau$ which is the simultaneous fixed point of $\overrightarrow{a}$ with respect to $\overrightarrow{E}$ in iGL, i.e.*

- *iGL $\vdash \tau(E_i) \leftrightarrow \tau(a_i)$ for every $1 \leq i \leq m$.*

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## Unification

- Unification problem (in propositional Logic L) asks for substitutions $\theta$ which unify $A$, i.e. $L \vdash \theta(A)$.
- More ambitiously: describe the set of all unifiers for $A$.
- $\theta \leq \gamma$ iff there is some $\lambda$ s.t. $L \vdash \theta(x) \leftrightarrow \lambda\gamma(x)$.
- Classical logic: every unifiable proposition has a most general unifier.

If $\theta$ is a unifier of $A$ then $\chi_\theta$ is a most general one:

$$\chi_\theta(x) := (A \wedge x) \vee (\neg A \wedge \theta(x))$$

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$\chi_\theta$ is a unifier indeed:

$$A \vdash \chi_\theta(A) \leftrightarrow \top \quad \text{and} \quad \neg A \vdash \chi_\theta(A) \leftrightarrow \top.$$

$\chi_\theta$ is more general than every other unifier $\gamma$:

$$A \vdash \chi_\theta(x) \leftrightarrow x \implies \gamma(A) \vdash \gamma\chi_\theta(x) \leftrightarrow \gamma(x)$$

### Definition

$A$ is called projective (in $\mathsf{L}$) if there is some unifier $\theta$ for $A$ s.t. for every $x \in \mathsf{var}$:

$$A \vdash_\mathsf{L} \theta(x) \leftrightarrow x$$

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- $x \vee \neg x$ does not have a most general unifier in IPC.

- Ghilardi [1999] answered to the unification problem for $\mathsf{L} = \mathsf{IPC}$ and $\mathsf{par} = \emptyset$ (Elementary unification or E-unification).

- Ghilardi [1999] first characterized projectives via Kripke semantics.

- Then with the aid of projective approximations he proved that IPC is finitary, i.e. every unifiable $A$ has a finite set of unifiers which are more general than every unifier of $A$.

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# Projectivity: relativised

- Instead of unification ($\top$-fication) we consider $\Gamma$-fication for $\Gamma \subseteq \mathcal{L}_0(\mathsf{par})$.

- This means that we ask for all $\theta$'s such that $\mathsf{L} \vdash \theta(A) \in \Gamma$, i.e. $\mathsf{L} \vdash \theta(A) \leftrightarrow E$ for some $E \in \Gamma$.

- In this setting, we say that $A$ is $\Gamma$-projective iff there is a $\Gamma$-fier $\theta$ for $A$ which is projective:

$$A \vdash_{\mathsf{L}} \theta(x) \leftrightarrow x$$

- $\downarrow\Gamma :=$ the set of all $\Gamma$-projective propositions.

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In the first of two consecutive manuscripts on provability logic of HA we considered the case $\Gamma = \mathsf{NNIL}(\mathsf{par})$ and $\mathsf{L} = \mathsf{IPC}$. We followed Ghilardi [1999] to

- characterize $\mathsf{NNIL}(\mathsf{par})$-projectivity via Kripke semantics,
- and then for a given $A$, compute a finite $\mathsf{NNIL}(\mathsf{par})$-projective approximation.

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## Admissible rules

- The problem of admissibility (Friedman 1975) asks for the characterization and decidability of all inference rules $A/B$ which are admissible to the logic L, i.e. for every substitution $\theta$ if we have $\mathsf{L} \vdash \theta(A)$ then $\mathsf{L} \vdash \theta(B)$.

- The classical case is trivial: $A/B$ is admissible iff $A \to B$ is derivable.

- $\neg x \to (y \vee z)/(\neg x \to y) \vee (\neg x \to z)$ is admissible to IPC. [Harrop, 1960]

- Rybakov [1987] showed that admissibility for IPC is decidable.

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## Admissible rules of IPC

- de Jongh and Visser provided a base for all known admissible rules of IPC and conjectured it to be complete.

- Iemhoff [2001b] with the aid of [Ghilardi, 1999] proved the completeness of the base.

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## Relative admissibility

- In the first manuscript, we considered a relative version of admissibility.

- We say that $A/B$ is admissible relative in $\Gamma$ if

$$\forall E \in \Gamma \; \forall \theta \; ( \vdash \theta(E \to A) \Longrightarrow \vdash \theta(E \to B)).$$

- Following the tools and methods in [Iemhoff, 2001b] we found a base for the admissibility relative in NNIL(par).

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# NNIL(par)-projective approximation

◀ ($\downarrow$N($\square$)$^\vee$, iGL) rsdc

### Theorem

*Given $A \in \mathcal{L}_0$, there is a finite set $\Pi \subseteq \downarrow$N(par) such that*

1. IPC $\vdash \bigvee \Pi \to A$.

2. $[\![$IPC, par$]\!] \vdash A \rhd \bigvee \Pi$. ▶

3. $\Pi$ *is computable as a function of $A$.*

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$[\![ \mathsf{T}, \Delta ]\!]$ has following axioms and rules:

Ax : $\quad A \rhd B$, for every $\mathsf{T} \vdash A \to B$.

$\mathsf{V}(\Delta)$ : $\quad B \to C \rhd \bigvee_{i=1}^{n+m} \{B\}_\Delta(E_i)$, in which $B = \bigwedge_{i=1}^{n}(E_i \to F_i)$
$\quad$ and $C = \bigvee_{i=n+1}^{n+m} E_i$, and

$$\frac{A \rhd B \quad A \rhd C}{A \rhd B \wedge C} \text{ Conj} \qquad\qquad \frac{A \rhd B \quad B \rhd C}{A \rhd C} \text{ Cut}$$

$$\frac{B \rhd A \quad C \rhd A}{B \vee C \rhd A} \text{ Disj} \qquad\qquad \frac{A \rhd B \quad (C \in \Delta)}{C \to A \rhd C \to B} \text{ Mont}(\Delta)$$

$$\{A\}_\Delta(B) := \begin{cases} B & : B \in \Delta \\ A \to B & : \text{otherwise} \end{cases}$$

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## Elevating projectivity to the modal language I

Let $A \in \mathcal{L}_\Box$ and $\Gamma \subseteq \mathcal{L}_0(\mathsf{parb})$. A substitution $\theta$ is called
*A-projective* (in T) if

For all atomic $a$ we have $\quad \mathsf{T} \vdash A \to (a \leftrightarrow \theta(a)).$ (3.1)

A substitution $\theta$, is a $\Gamma$-*fier* for $A \in \mathcal{L}_\Box$ (notation $A \xrightarrow[\mathsf{T}]{\theta} \Gamma$), if

$\mathsf{T} \vdash \hat{\theta}(A) \in \Gamma$ i.e. $\hat{\theta}(A)$ is T-equivalent to some $A' \in \Gamma$.

$\theta$ is a unifier for $A$ if it is $\{\top\}$-fier for $A$.

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## Elevating projectivity to the modal language II

- We say that a substitution $\theta$ projects $A$ to $\Gamma$ in T (notation: $A \overset{\theta}{\underset{\mathsf{T}}{\rightarrowtail\!\!\!\rightarrow}} \Gamma$) if $\theta$ is $A$-projective in T and $A \overset{\theta}{\underset{\mathsf{T}}{\twoheadrightarrow}} \Gamma$.

- We say that $A$ is $\Gamma$-projective in T if there is some $\theta$ such that $A \overset{\theta}{\underset{\mathsf{T}}{\rightarrowtail\!\!\!\rightarrow}} \Gamma$.

- $\overset{\mathsf{T}}{\downarrow}\Gamma$ indicates the set of all propositions which are $\Gamma$-projective in T.

- $A$ is projective, if it is $\{\top\}$-projective.

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## Some notations on sets of propositions

- We write $X_1 \ldots X_n$ for $X_1 \cap \ldots \cap X_n$.
- $\Gamma^\vee := \{\bigvee \Delta : \Delta \subseteq_{\mathrm{fin}} \Gamma \text{ and } \Delta \neq \emptyset\}$.
- $\Gamma(X) := \Gamma \cap \mathcal{L}_0(X)$ and $\Gamma(\square) := \Gamma(\mathsf{parb})$.
- $\downarrow^{\mathsf{T}} \Gamma :=$ the set of all $\Gamma$-projective propositions in the logic $\mathsf{T}$.
- $(.)^\vee$ has the lowest precedence and $\downarrow(.)$ has the second lowest precedence. This means that

$$\downarrow\mathsf{SN}(\square)^\vee := (\downarrow(\mathsf{SN}(\square)))^\vee \quad \text{and} \quad \mathsf{C}\downarrow\mathsf{SN}(\square)^\vee := (\mathsf{C}(\downarrow(\mathsf{SN}(\square))))^\vee.$$

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# Definitions of admissibility and preservativity

$$A \mathrel{\vdash\!\!\!\sim}^{\scriptscriptstyle T}_{\scriptscriptstyle \Gamma} B \text{ iff for every substitution } \theta \text{ and } C \in \Gamma:$$
$$\mathsf{T} \vdash \hat{\theta}(C \to A) \Rightarrow \mathsf{T} \vdash \hat{\theta}(C \to B).$$

$$A \mathrel{\approx\!\!\!\mid}^{\scriptscriptstyle T}_{\scriptscriptstyle \Gamma} B \quad \text{ iff } \quad \forall\, E \in \Gamma(\mathsf{T} \vdash E \to A \Rightarrow \mathsf{T} \vdash E \to B).$$

$\hat{\theta}$ is same as $\theta$ on the non-modal language and $\hat{\theta}(\Box B) := \Box B$.

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By definition it can be inferred that $A \mathrel{\vdash^{\scriptscriptstyle\mathsf{T}}_{\scriptscriptstyle\Gamma}} B$ implies $A \mathrel{\approx^{\scriptscriptstyle\mathsf{T}}_{\scriptscriptstyle\Gamma}} B$, however the converse may not hold. As a counterexample let $A$ and $B$ two different variables and $\Gamma := \{\top\}$ and $\mathsf{T} = \mathsf{IPC}$. Then we have $A \mathrel{\approx^{\scriptscriptstyle\mathsf{T}}_{\scriptscriptstyle\Gamma}} B$ and not $A \mathrel{\vdash^{\scriptscriptstyle\mathsf{T}}_{\scriptscriptstyle\Gamma}} B$.

### Theorem

*Let $\mathsf{T}$ be a logic which is closed under outer substitutions. Then $A \mathrel{\vdash^{\scriptscriptstyle\mathsf{T}}_{\scriptscriptstyle\Gamma}} B$ implies $A \mathrel{\approx^{\scriptscriptstyle\mathsf{T}}_{\scriptscriptstyle\downarrow\Gamma}} B$.*

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## Some more notations

- $\mathsf{P}^\mathsf{T} := \{A : \mathsf{T} \vdash A \to B \vee C \Rightarrow \mathsf{T} \vdash A \to B \text{ or } \mathsf{T} \vdash A \to C\}.$

- $\Gamma^\vee := \{\bigvee \Delta : \emptyset \neq \Delta \subseteq_{\text{fin}} \Gamma\}.$

We may omit $\mathsf{T}$ from notations $\mathsf{P}^\mathsf{T}$ and $\mathsf{C}^\mathsf{T}$.

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## [T]

◂ SN(□)-rsdc    ◂ Projec-Res

Given a logic $\mathsf{T}$, the logic $[\mathsf{T}]$ proves statements $A \rhd B$ for $A$ and $B$ in the language of $\mathsf{T}$ and has the following axioms and rules:

**Aximos**

Ax :    $A \rhd B$, for every $\mathsf{T} \vdash A \rightarrow B$.

**Rules**

$$\frac{A \rhd B \qquad A \rhd C}{A \rhd B \wedge C} \text{ Conj} \qquad\qquad \frac{A \rhd B \qquad B \rhd C}{A \rhd C} \text{ Cut}$$

These axioms and rule are not interesting, because $[\mathsf{T}] \vdash A \rhd B$ iff $\mathsf{T} \vdash A \rightarrow B$.

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# Extra axioms for preservativity and admissibility

Le: $\quad A \rhd \square A$ for every $A \in \mathcal{L}_\square$.

Le$^-$: $\quad A \rhd \square A$ for every $A \in \mathcal{L}_0(\mathsf{parb})$.

A: $\quad A \rhd \hat{\theta}(A)$, for every substitution $\theta$.

V($\Delta$): $\quad B \to C \rhd \bigvee_{i=1}^{n+m}\{B\}_\Delta(E_i)$, in which $B = \bigwedge_{i=1}^{n}(E_i \to F_i)$
$\quad\quad$ and $C = \bigvee_{i=n+1}^{n+m} E_i$, and

$$\{A\}_\Delta(B) := \begin{cases} B & : B \in \Delta \\ A \to B & : \text{otherwise} \end{cases}$$

$$\frac{B \rhd A \quad C \rhd A}{B \vee C \rhd A} \text{ Disj} \quad\quad \frac{A \rhd B \quad (C \in \Delta)}{C \to A \rhd C \to B} \text{ Mont}(\Delta)$$

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$$\llbracket \mathsf{T}, \Delta \rrbracket := [\mathsf{T}] + \text{Disj} + \text{Mont}(\Delta) + \mathsf{V}(\Delta),$$

$$\llbracket \mathsf{T}, \Delta \rrbracket \mathsf{Le} := \llbracket \mathsf{T}, \Delta \rrbracket + \mathsf{Le} \quad \text{and} \quad \llbracket \mathsf{T}, \Delta \rrbracket \mathsf{Le}^- := \llbracket \mathsf{T}, \Delta \rrbracket + \mathsf{Le}^-.$$

### Lemma

$\mathsf{T} \subseteq \mathsf{T}'$ and $\Delta \subseteq \Delta'$ implies $\llbracket \mathsf{T}, \Delta \rrbracket \subseteq \llbracket \mathsf{T}', \Delta' \rrbracket$.

### Lemma

$\models^{\mathsf{T}}_{\Gamma} = \models^{\mathsf{T}}_{\Gamma^\approx}$ and $\vdash^{\mathsf{T}}_{\Gamma} = \vdash^{\mathsf{T}}_{\Gamma^\vee}$.

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## Intuitionistic submodel property

Given two Kripke models $\mathcal{K} = (W, \preccurlyeq, \sqsubset, V)$ and $\mathcal{K}' = (W', \preccurlyeq', \sqsubset', V')$, we say that $\mathcal{K}'$ is an intuitionistic submodel of $\mathcal{K}$ (notation $\mathcal{K}' \leq \mathcal{K}$) iff $W = W'$, $\sqsubset = \sqsubset'$, $V = V'$ and $\preccurlyeq' \subseteq \preccurlyeq$. A class $\mathscr{K}$ of Kripke models has intuitionistic submodel property, if $\mathcal{K}' \leq \mathcal{K} \in \mathscr{K}$ implies $\mathcal{K}' \in \mathscr{K}$. A modal logic T is said to have intuitionistic submodel property iff it is sound and complete for some class $\mathscr{K}$ of good Kripke models with intuitionistic submodel property.

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## General soundness: preservativity

### Theorem (**Soundness**)

[T] *is sound for preservativity interpretations, i.e.* $[\mathsf{T}] \vdash A \rhd B$ *implies* $A \underset{\Gamma}{\overset{\scriptscriptstyle\approx}{\vDash}} B$ *for every set* $\Gamma$ *of propositions and every logic* $\mathsf{T}$. *Moreover*

- *if* $\Gamma$ *is* $\mathsf{T}$-*complete, then* Le *is sound,*
- *if* $\Gamma$ *is* $\mathsf{T}$-*prime, then Disj is also sound,*
- *if* $\Gamma$ *is closed under* $\Delta$-*conjunctions, then* $\mathrm{Mont}(\Delta)$ *is sound.*
- *if* $\mathsf{T}$ *has intuitionistic submodel property and* $\Gamma \subseteq \mathsf{NNIL}$ *and* $\Delta \subseteq$ atomb *then* $\mathsf{V}(\Delta)$ *is sound.*
- *if* $\Gamma \subseteq \mathcal{L}_0(\mathsf{parb})$ *and* $\mathsf{T}$ *is closed under outer substitutions, then* A *is also sound.*

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# General soundness: admissibility

### Theorem (**Soundness**)

[T] *is sound for admissibility interpretations, i.e.* $[\mathsf{T}] \vdash A \triangleright B$
*implies* $A \mathrel{\mathop{\vdash}\limits_{\Gamma}^{\mathrm{T}}} B$ *for every set* $\Gamma$ *of propositions and every logic* T
*which is closed under outer substitutions. Moreover*

- *if* $\Gamma$ *is* T*-complete, then* $\mathsf{Le}^-$ *is sound,*
- *if* $\Gamma$ *is* T*-prime, then Disj is also sound.*
- *if* $\Gamma$ *is closed under outer substitutions of* $\Delta$*-conjunctions, i.e.* $A \in \Gamma$ *and* $B \in \Delta$ *implies* $A \wedge \hat{\theta}(B) \in \Gamma$ *(up to* T*-provable equivalence relation), then* $\mathrm{Mont}(\Delta)$ *is sound.*
- *if* T *has intuitionistic submodel property and* $\Gamma \subseteq \mathsf{NNIL}$ *and* $\Delta \subseteq \mathsf{parb}$ *then* $\mathsf{V}(\Delta)$ *is sound.*

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# On Provability Logic of HA

Mojtaba Mojtahedi (University of Tehran)

University of Tehran

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# Greatest lower bound (glb)

- $B$ is a $(\Gamma, \mathsf{T})$-lb for $A$ if:
  1. $B \in \Gamma$,
  2. $\mathsf{T} \vdash B \to A$.

- $B$ is the $(\Gamma, \mathsf{T})$-glb for $A$, if for every $(\Gamma, \mathsf{T})$-lb $B'$ for $A$ we have $\mathsf{T} \vdash B' \to B$.

- Up to $\mathsf{T}$-provable equivalence relation, such glb is unique and we annotate it as $\lfloor A \rfloor_\Gamma^\mathsf{T}$.

- $(\Gamma, \mathsf{T})$ is downward compact, if every $A \in \mathcal{L}_\Box$ has a $(\Gamma, \mathsf{T})$-glb $\lfloor A \rfloor_\Gamma^\mathsf{T}$.

- If $\lfloor A \rfloor_\Gamma^\mathsf{T}$ can be effectively computed, we say that $(\Gamma, \mathsf{T})$ is recursively downward compact.

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◀ (SN, iGLC$_a$) is rsdc

### Theorem (Visser [2002])

$(\mathsf{NNIL}, \mathsf{IPC})$ *is recursively downward compact.*

$\lfloor A \rfloor^{\mathsf{IPC}}_{\mathsf{NNIL}}$ is named $A^*$ in [Visser, 2002], the so called Visser's $\mathsf{NNIL}$ algorithm.

### Question

*One may similarly define the notion of least upper bounds and upward compactness. Does downward compactness imply upward compactness?*

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# $(\Gamma, \mathsf{T})$-glb and $\overset{\mathsf{T}}{\underset{\Gamma}{\not\approx}}$

## Theorem

$B$ is the $(\Gamma, \mathsf{T})$-glb for $A$ iff

- $B \in \Gamma$,
- $\mathsf{T} \vdash B \to A$,
- $A \overset{\mathsf{T}}{\underset{\Gamma}{\not\approx}} B$.

Hence we have $A \overset{\mathsf{T}}{\underset{\Gamma}{\not\approx}} \lfloor A \rfloor_\Gamma^\mathsf{T}$.

## Corollary

If $\lfloor A \rfloor_\Gamma^\mathsf{T}$ exists, then for every $B \in \mathcal{L}_\square$ we have

$$\mathsf{T} \vdash \lfloor A \rfloor_\Gamma^\mathsf{T} \to B \quad \textit{iff} \quad A \overset{\mathsf{T}}{\underset{\Gamma}{\not\approx}} B.$$

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### Question

*The glb may be expressed via preservativity relation $\underset{\Gamma}{\overset{\mathsf{T}}{\succcurlyeq}}$. One may think of its twin sister which best suites for lub's:*

$$A \; \underset{\Gamma}{\overset{*\mathsf{T}}{\succcurlyeq}} \; B \quad \text{iff} \quad \forall E \in \Gamma(\mathsf{T} \vdash A \to E \Rightarrow \mathsf{T} \vdash B \to E).$$

*We ask for an axiomatization for $\underset{\Gamma}{\overset{*\mathsf{T}}{\succcurlyeq}}$ when we let $\mathsf{T} = \mathsf{IPC}$ and $\Gamma = \mathsf{NNIL}$.*

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## Normal forms

Define $\Gamma\text{-}\mathsf{NF}_0$ as the set of propositions $B \in \mathcal{L}_\Box$ with either $B \in \Gamma$ or $\boxdot B \in \Gamma$. Then define the set $\Gamma\text{-}\mathsf{NF}$ of propositions in $\Gamma$-Normal Form as follows:

$$\Gamma\text{-}\mathsf{NF} := \{A \in \mathcal{L}_\Box : \forall \Box B \in \mathsf{sub}(A) \ B \in \Gamma\text{-}\mathsf{NF}_0\}.$$

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## Iterating glb's nested inside □

<span>◄ General Completeness</span>

We say that $(\Gamma, \mathsf{T})$ is (recursively) *strong* downward compact, if it is (recursively) downward compact and for every $\Box B \in \mathsf{sub}(\lfloor A \rfloor_\Gamma^\mathsf{T})$ either we have $\Box B \in \mathsf{sub}(A)$ or $B \in \Gamma\text{-}\mathsf{NF}_0$. We also inductively define $\lfloor A \rfloor_\Gamma^\mathsf{T}$:

- $\lfloor a \rfloor_\Gamma^\mathsf{T} = a$ for every atomic $a$.
- $\lfloor\ \rfloor_\Gamma^\mathsf{T}$ commutes with $\{\vee, \wedge, \to\}$.
- $\lfloor \Box A \rfloor_\Gamma^\mathsf{T} := \Box \lfloor \lfloor A \rfloor_\Gamma^\mathsf{T} \rfloor_\Gamma^\mathsf{T}$. <span>► H($\Gamma$, T)</span>.

### Lemma

*If $(\Gamma, \mathsf{T})$ is strong downward compact and $\mathsf{T} \supseteq \mathsf{iK4}$, then for every $A \in \mathcal{L}_\Box$ we have $\lfloor A \rfloor_\Gamma^\mathsf{T} \in \Gamma\text{-}\mathsf{NF}$ and $\mathsf{H}(\Gamma, \mathsf{T}) \vdash_\mathsf{T} A \leftrightarrow \lfloor A \rfloor_\Gamma^\mathsf{T}$.*

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## Extension property

A class $\mathcal{M}$ of rooted Kripke models is said to has *extension property* if for every finite set $\mathcal{K} \subseteq \mathcal{M}$ there is some finite set of rooted Kripke models $\mathcal{K}'$ such that a variant of $\sum(\mathcal{K}, \mathcal{K}')$ belongs to $\mathcal{M}$.

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## Prime factorization

Before we continue with the axiomatization and decidability of several preservativities, let us see some preliminaries.

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**Theorem**

*Let* T *has extension property. Then*

- $N(\square) = PN(\square)^\vee$ *and* $SN(\square) = SPN(\square)^\vee$.

- $N = PN^\vee$ *and* $SN = SPN^\vee$, *whenever* $T \supseteq iK4C_a$.

---

**Corollary**

$\underset{\mathsf{SN}(\square)}{\mathrel{\vrule width0pt}\approx^{\mathsf{T}}} \; = \underset{\mathsf{SPN}(\square)}{\mathrel{\vrule width0pt}\approx^{\mathsf{T}}} \; and \; \vdash^{\mathsf{T}}_{\mathsf{SN}(\square)} \; = \vdash^{\mathsf{T}}_{\mathsf{SPN}(\square)} \; and \; \vdash^{\mathsf{T}}_{\mathsf{N}(\square)} \; = \vdash^{\mathsf{T}}_{\mathsf{PN}(\square)} \; and \; if$
$T \supseteq iK4C_a \; then \; \underset{\mathsf{SN}}{\mathrel{\vrule width0pt}\approx^{\mathsf{T}}} \; = \underset{\mathsf{SPN}}{\mathrel{\vrule width0pt}\approx^{\mathsf{T}}} .$

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## Theorem

$A^{\mathsf{h}} := (A^*)^\square = \lfloor A \rfloor_{\mathsf{SN}}^{\mathsf{iGLC_a}}$ and hence $(\mathsf{SN}, \mathsf{iGLC_a})$ is recursively strong downward compact. Moreover $[\![\mathsf{iGLC_a}, \mathsf{atomb}]\!]\mathsf{Le} \vdash A \rhd A^{\mathsf{h}}$.

## Proof.

Derived by ▸ rdc of (NNIL, IPC) from Visser [2002]. ☐

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$$\text{Ax:} \quad A \triangleright B, \text{ for every iGLC}_a \vdash A \to B.$$

$$\text{Le:} \quad A \triangleright \square A \text{ for every } A \in \mathcal{L}_\square.$$

$$\text{V(atomb):} \quad B \to C \triangleright \bigvee_{i=1}^{n+m}\{B\}_{\mathsf{atomb}}(E_i), \text{ in which}$$
$$B = \bigwedge_{i=1}^{n}(E_i \to F_i) \text{ and } C = \bigvee_{i=n+1}^{n+m} E_i, \text{ and}$$

$$\{A\}_{\mathsf{atomb}}(B) := \begin{cases} B & : B \in \mathsf{atomb} \\ A \to B & : \text{otherwise} \end{cases}$$

$$\frac{A \triangleright B \qquad A \triangleright C}{A \triangleright B \wedge C} \text{ Conj} \qquad\qquad \frac{A \triangleright B \qquad B \triangleright C}{A \triangleright C} \text{ Cut}$$

$$\frac{B \triangleright A \qquad C \triangleright A}{B \vee C \triangleright A} \text{ Disj} \qquad\qquad \frac{A \triangleright B \qquad (C \in \mathsf{atomb})}{C \to A \triangleright C \to B}$$

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## Theorem

$$[\![\mathsf{iGLC_a}, \mathsf{atomb}]\!]\mathsf{Le} = \mathrel{\approx\!\!\!\!\approx}_{\mathsf{SN}}^{\mathsf{iGLC_a}} = \mathrel{\approx\!\!\!\!\approx}_{\mathsf{SN}^\vee}^{\mathsf{iGLC_a}} = \mathrel{\approx\!\!\!\!\approx}_{\mathsf{SPN}}^{\mathsf{iGLC_a}} = \mathrel{\approx\!\!\!\!\approx}_{\mathsf{SPN}^\vee}^{\mathsf{iGLC_a}} .$$

*Moreover all above relations are decidable.*

## Proof.

▸ Prime factorization and ▸ $\approx\, = \,\approx$ imply $\mathrel{\approx\!\!\!\!\approx}_{\mathsf{SN}}^{\mathsf{iGLC_a}} = \mathrel{\approx\!\!\!\!\approx}_{\mathsf{SN}^\vee}^{\mathsf{iGLC_a}} = \mathrel{\approx\!\!\!\!\approx}_{\mathsf{SPN}}^{\mathsf{iGLC_a}} = \mathrel{\approx\!\!\!\!\approx}_{\mathsf{SPN}^\vee}^{\mathsf{iGLC_a}}$ .

▸ General soundness implies $[\![\mathsf{iGLC_a}, \mathsf{atomb}]\!]\mathsf{Le} \subseteq \mathrel{\approx\!\!\!\!\approx}_{\mathsf{SPN}}^{\mathsf{iGLC_a}}$ .

To show $\mathrel{\approx\!\!\!\!\approx}_{\mathsf{SN}}^{\mathsf{iGLC_a}} \subseteq [\![\mathsf{iGLC_a}, \mathsf{atomb}]\!]\mathsf{Le}$, let $A \mathrel{\approx\!\!\!\!\approx}_{\mathsf{SN}}^{\mathsf{iGLC_a}} B$. Then $A^{\mathsf{h}} \to B$ and hence $[\![\mathsf{iGLC_a}, \mathsf{atomb}]\!]\mathsf{Le} \vdash A^{\mathsf{h}} \rhd B$. Since $[\![\mathsf{iGLC_a}, \mathsf{atomb}]\!]\mathsf{Le} \vdash A \rhd A^{\mathsf{h}}$, Cut implies desired result. $\qquad \square$

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$\blacktriangleleft \downarrow SN(\Box)^\vee$-rsdc

### Theorem

$(\downarrow N(\Box)^\vee, iGL)$ *is recursively strong downward compact.*
*Moreover* $[\![iGL, parb]\!] \vdash A \rhd \lfloor A \rfloor_{\downarrow N(\Box)^\vee}^{iGL}$.

### Proof sketch.

Given $A$, one must treat outer occurrences of $\Box$'s as parameters, and then $\bigvee \Pi$ in $\blacktriangleright$ NNIL(par)-projective approximation will work as $\lfloor A \rfloor_{\downarrow N(\Box)^\vee}^{iGL}$. $\qquad\qquad\qquad\qquad\qquad\qquad\Box$

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$\text{Ax}:$    $A \rhd B$, for every $\text{iGLC}_a \vdash A \to B$.

$\text{V(parb)}:$    $B \to C \rhd \bigvee_{i=1}^{n+m} \{B\}_{\text{parb}}(E_i)$, in which $B = \bigwedge_{i=1}^{n}(E_i \to F_i)$
        and $C = \bigvee_{i=n+1}^{n+m} E_i$, and

$$\{A\}_{\text{parb}}(B) := \begin{cases} B & : B \in \text{parb} \\ A \to B & : \text{otherwise} \end{cases}$$

$$\dfrac{A \rhd B \quad A \rhd C}{A \rhd B \wedge C} \text{ Conj} \qquad \qquad \dfrac{A \rhd B \quad B \rhd C}{A \rhd C} \text{ Cut}$$

$$\dfrac{B \rhd A \quad C \rhd A}{B \vee C \rhd A} \text{ Disj} \qquad \qquad \dfrac{A \rhd B \quad (C \in \text{parb})}{C \to A \rhd C \to B}$$

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## Theorem

*All are decidable:*

$$[\![\mathsf{iGL}, \mathsf{parb}]\!] = \overset{\mathsf{iGL}}{\underset{\downarrow \mathsf{N}(\Box)}{\approx}} = \overset{\mathsf{iGL}}{\underset{\downarrow \mathsf{N}(\Box)^\vee}{\approx}} = \overset{\mathsf{iGL}}{\underset{\downarrow \mathsf{PN}(\Box)}{\approx}} = \overset{\mathsf{iGL}}{\underset{\downarrow \mathsf{PN}(\Box)^\vee}{\approx}} =$$
$$= \overset{\mathsf{iGL}}{\underset{\mathsf{N}(\Box)}{\vdash}} = \overset{\mathsf{iGL}}{\underset{\mathsf{N}(\Box)^\vee}{\vdash}} = \overset{\mathsf{iGL}}{\underset{\mathsf{PN}(\Box)}{\vdash}} = \overset{\mathsf{iGL}}{\underset{\mathsf{PN}(\Box)^\vee}{\vdash}} \, .$$

## Proof sketch.

▸ Prime factorization and ▸ $\vdash = \vdash$ imply $\overset{\mathsf{iGL}}{\underset{\downarrow \mathsf{N}(\Box)^\vee}{\approx}} = \overset{\mathsf{iGL}}{\underset{\downarrow \mathsf{PN}(\Box)}{\approx}} = \overset{\mathsf{iGL}}{\underset{\downarrow \mathsf{PN}(\Box)^\vee}{\approx}}$ and
$\overset{\mathsf{iGL}}{\underset{\mathsf{N}(\Box)}{\vdash}} = \overset{\mathsf{iGL}}{\underset{\mathsf{N}(\Box)^\vee}{\vdash}} = \overset{\mathsf{iGL}}{\underset{\mathsf{PN}(\Box)}{\vdash}} = \overset{\mathsf{iGL}}{\underset{\mathsf{PN}(\Box)^\vee}{\vdash}}$. Moreover ▸ implies $\overset{\mathsf{iGL}}{\underset{\mathsf{N}(\Box)}{\vdash}} \subseteq \overset{\mathsf{iGL}}{\underset{\downarrow \mathsf{N}(\Box)}{\approx}}$.
▸ General soundenss implies $[\![\mathsf{iGL}, \mathsf{parb}]\!] \subseteq \overset{\mathsf{iGL}}{\underset{\mathsf{N}(\Box)}{\vdash}}$. To show
$\overset{\mathsf{iGL}}{\underset{\downarrow \mathsf{N}(\Box)^\vee}{\approx}} \subseteq [\![\mathsf{iGL}, \mathsf{parb}]\!]$, let $A \overset{\mathsf{iGL}}{\underset{\downarrow \mathsf{N}(\Box)}{\approx}} B$. Hence $\mathsf{iGL} \vdash \lfloor A \rfloor^{\mathsf{iGL}}_{\downarrow \mathsf{N}(\Box)^\vee} \to B$
and thus $[\![\mathsf{iGL}, \mathsf{parb}]\!] \vdash \lfloor A \rfloor^{\mathsf{iGL}}_{\downarrow \mathsf{N}(\Box)^\vee} \rhd B$. Since
$[\![\mathsf{iGL}, \mathsf{parb}]\!] \vdash A \rhd \lfloor A \rfloor^{\mathsf{iGL}}_{\downarrow \mathsf{N}(\Box)^\vee}$ we are done. □

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### Theorem

$(\downarrow\mathsf{SN}(\square)^\vee, \mathsf{iGL})$ *is recursively strong downward compact.*
*Moreover* $[\![\mathsf{iGL}, \mathsf{parb}]\!]\mathsf{Le}^- \vdash A \triangleright \lfloor A \rfloor_{\downarrow\mathsf{SN}(\square)^\vee}^{\mathsf{iGL}}$.

### Proof sketch.

Given $A$, one first compute $\lfloor A \rfloor_{\downarrow\mathsf{N}(\square)^\vee}^{\mathsf{iGL}}$ 〔▶〕. Let $B$ is its
$\mathsf{N}(\square)^\vee$-projection and define $\lfloor A \rfloor_{\downarrow\mathsf{SN}(\square)^\vee}^{\mathsf{iGL}} := \lfloor A \rfloor_{\downarrow\mathsf{N}(\square)^\vee}^{\mathsf{iGL}} \wedge B^\square$. □

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$\mathsf{Ax}$: $A \triangleright B$, for every $\mathsf{iGLC_a} \vdash A \to B$.

$\mathsf{Le}^-$: $A \triangleright \square A$ for every $A \in \mathcal{L}_0(\mathsf{parb})$.

$\mathsf{V(parb)}$: $B \to C \triangleright \bigvee_{i=1}^{n+m} \{B\}_{\mathsf{parb}}(E_i)$, in which $B = \bigwedge_{i=1}^{n}(E_i \to F_i)$
and $C = \bigvee_{i=n+1}^{n+m} E_i$, and

$$\{A\}_{\mathsf{parb}}(B) := \begin{cases} B & : B \in \mathsf{parb} \\ A \to B & : \text{otherwise} \end{cases}$$

$$\dfrac{A \triangleright B \quad A \triangleright C}{A \triangleright B \wedge C} \text{ Conj} \qquad\qquad \dfrac{A \triangleright B \quad B \triangleright C}{A \triangleright C} \text{ Cut}$$

$$\dfrac{B \triangleright A \quad C \triangleright A}{B \vee C \triangleright A} \text{ Disj} \qquad\qquad \dfrac{A \triangleright B \quad (C \in \mathsf{parb})}{C \to A \triangleright C \to B}$$

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## Theorem

$$[\![\mathsf{iGL}, \mathsf{parb}]\!]\mathsf{Le}^- = \underset{\downarrow\mathsf{SN}(\Box)}{\overset{\mathsf{iGL}}{\vDash}} = \underset{\downarrow\mathsf{SN}(\Box)^\vee}{\overset{\mathsf{iGL}}{\vDash}} = \underset{\downarrow\mathsf{SPN}(\Box)}{\overset{\mathsf{iGL}}{\vDash}} = \underset{\downarrow\mathsf{SPN}(\Box)^\vee}{\overset{\mathsf{iGL}}{\vDash}} =$$
$$= \underset{\mathsf{SN}(\Box)}{\overset{\mathsf{iGL}}{\vdash}} = \underset{\mathsf{SN}(\Box)^\vee}{\overset{\mathsf{iGL}}{\vdash}} = \underset{\mathsf{SPN}(\Box)}{\overset{\mathsf{iGL}}{\vdash}} = \underset{\mathsf{SPN}(\Box)^\vee}{\overset{\mathsf{iGL}}{\vdash}} \ .$$

## Proof sketch.

▸ Prime factorization and ▸ $\vDash = \vDash\vee$ imply
$\underset{\downarrow\mathsf{SN}(\Box)}{\overset{\mathsf{iGL}}{\vDash}} = \underset{\downarrow\mathsf{SN}(\Box)^\vee}{\overset{\mathsf{iGL}}{\vDash}} = \underset{\downarrow\mathsf{SPN}(\Box)}{\overset{\mathsf{iGL}}{\vDash}} = \underset{\downarrow\mathsf{SPN}(\Box)^\vee}{\overset{\mathsf{iGL}}{\vDash}}$ and
$\underset{\mathsf{SN}(\Box)}{\overset{\mathsf{iGL}}{\vdash}} = \underset{\mathsf{SN}(\Box)^\vee}{\overset{\mathsf{iGL}}{\vdash}} = \underset{\mathsf{SPN}(\Box)}{\overset{\mathsf{iGL}}{\vdash}} = \underset{\mathsf{SN}(\Box)^\vee}{\overset{\mathsf{iGL}}{\vdash}}$ . Moreover ▸ implies
$\underset{\mathsf{SN}(\Box)}{\overset{\mathsf{iGL}}{\vdash}} \subseteq \underset{\downarrow\mathsf{SN}(\Box)}{\overset{\mathsf{iGL}}{\vDash}}$ . ▸ General soundness implies $[\![\mathsf{iGL}, \mathsf{parb}]\!]\mathsf{Le}^- \subseteq \underset{\mathsf{SN}(\Box)}{\overset{\mathsf{iGL}}{\vdash}}$ . To
show $\underset{\downarrow\mathsf{SN}(\Box)^\vee}{\overset{\mathsf{iGL}}{\vDash}} \subseteq [\![\mathsf{iGL}, \mathsf{parb}]\!]\mathsf{Le}^-$, let $A \underset{\downarrow\mathsf{SN}(\Box)}{\overset{\mathsf{iGL}}{\vDash}} B$. Hence
$\mathsf{iGL} \vdash \lfloor A \rfloor^{\mathsf{iGL}}_{\downarrow\mathsf{SN}(\Box)^\vee} \to B$ and thus $[\![\mathsf{iGL}, \mathsf{parb}]\!]\mathsf{Le}^- \vdash \lfloor A \rfloor^{\mathsf{iGL}}_{\downarrow\mathsf{SN}(\Box)^\vee} \rhd B$.
Since $[\![\mathsf{iGL}, \mathsf{parb}]\!]\mathsf{Le}^- \vdash A \rhd \lfloor A \rfloor^{\mathsf{iGL}}_{\downarrow\mathsf{SN}(\Box)^\vee}$ we are done. $\qquad\square$

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$(\mathsf{C}{\downarrow}\mathsf{SN}(\Box)^\vee, \mathsf{iGL})$ *is recursively strong downward compact.*
*Moreover* $[\![\mathsf{iGL}, \mathsf{parb}]\!]\mathsf{Le} \vdash A \rhd \lfloor A \rfloor_{\mathsf{C}{\downarrow}\mathsf{SN}(\Box)^\vee}^{\mathsf{iGL}}$.

### Proof sketch.

Given $A$, one first compute $\lfloor A \rfloor_{{\downarrow}\mathsf{SN}(\Box)^\vee}^{\mathsf{iGL}}$ ▶. Then define

$$\lfloor A \rfloor_{\mathsf{C}{\downarrow}\mathsf{SN}(\Box)^\vee}^{\mathsf{iGL}} := \boxdot \lfloor A \rfloor_{{\downarrow}\mathsf{SN}(\Box)^\vee}^{\mathsf{iGL}} \qquad \Box$$

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Ax : $\quad A \triangleright B$, for every $\mathsf{iGLC_a} \vdash A \to B$.

Le : $\quad A \triangleright \square A$ for every $A \in \mathcal{L}_\square$.

V(parb) : $\quad B \to C \triangleright \bigvee_{i=1}^{n+m} \{B\}_{\mathsf{parb}}(E_i)$, in which $B = \bigwedge_{i=1}^{n}(E_i \to F_i)$
  and $C = \bigvee_{i=n+1}^{n+m} E_i$, and

$$\{A\}_{\mathsf{parb}}(B) := \begin{cases} B & : B \in \mathsf{parb} \\ A \to B & : \text{otherwise} \end{cases}$$

$$\frac{A \triangleright B \quad A \triangleright C}{A \triangleright B \wedge C} \text{ Conj} \qquad \frac{A \triangleright B \quad B \triangleright C}{A \triangleright C} \text{ Cut}$$

$$\frac{B \triangleright A \quad C \triangleright A}{B \vee C \triangleright A} \text{ Disj} \qquad \frac{A \triangleright B \quad (C \in \mathsf{parb})}{C \to A \triangleright C \to B}$$

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◀ iGLH $\subseteq$ iPH

### Theorem

$$[\![\text{iGL, parb}]\!]\text{Le} = \mathop{\vDash}\limits_{C\downarrow SN(\Box)}^{iGL} \;=\; \mathop{\vDash}\limits_{C\downarrow SN(\Box)^\vee}^{iGL} \;=\; \mathop{\vDash}\limits_{C\downarrow SPN(\Box)}^{iGL} \;=\; \mathop{\vDash}\limits_{C\downarrow SPN(\Box)^\vee}^{iGL}$$

*Moreover all mentioned relations are decidable.*

### Proof.

▸ Prime factorization and ▸ $\vDash = \vDash$ imply
$\mathop{\vDash}\limits_{C\downarrow SN(\Box)}^{iGL} = \mathop{\vDash}\limits_{C\downarrow SN(\Box)^\vee}^{iGL} = \mathop{\vDash}\limits_{C\downarrow SPN(\Box)}^{iGL} = \mathop{\vDash}\limits_{C\downarrow SPN(\Box)^\vee}^{iGL}$ . ▸ General soundenss implies
$[\![\text{iGL, parb}]\!]\text{Le} \subseteq \mathop{\vDash}\limits_{C\downarrow SPN(\Box)}^{iGL}$ . To show $\mathop{\vDash}\limits_{C\downarrow SN(\Box)^\vee}^{iGL} \subseteq [\![\text{iGL, parb}]\!]\text{Le}$, let
$A \mathop{\vDash}\limits_{C\downarrow SN(\Box)}^{iGL} B$. Hence $\text{iGL} \vdash \lfloor A \rfloor_{C\downarrow SN(\Box)^\vee}^{iGL} \to B$ and thus
$[\![\text{iGL, parb}]\!]\text{Le} \vdash \lfloor A \rfloor_{C\downarrow SN(\Box)^\vee}^{iGL} \rhd B$. Since
$[\![\text{iGL, parb}]\!]\text{Le} \vdash A \rhd \lfloor A \rfloor_{C\downarrow SN(\Box)^\vee}^{iGL}$ we are done. $\qquad\square$

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### Theorem

$(\mathsf{SN}(\square), \mathsf{iGL})$ *is recursively strong downward compact. Moreover*
$[\![\mathsf{iGL}, \mathsf{parb}]\!]\mathsf{LeA} \vdash A \triangleright \lfloor A \rfloor_{\mathsf{SN}(\square)}^{\mathsf{iGL}}$.

### A flavour of proof.

In the computation of $\lfloor A \rfloor_{\mathsf{SN}(\square)}^{\mathsf{iGL}}$ is not just an add on for the Visser's $\mathsf{NNIL}$-algorithm. One must go inside that algorithm and make some additional instruction.

$x \to B$ is approximated by $\lfloor B[\hat{x} : \top] \rfloor_{\mathsf{SN}(\square)}^{\mathsf{iGL}}$.

$B \to x$ is approximated by $\lfloor \neg B \rfloor_{\mathsf{SN}(\square)}^{\mathsf{iGL}}$. $\qquad \square$

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Ax :     $A \rhd B$, for every $\mathsf{iGLC_a} \vdash A \to B$.

Le:     $A \rhd \Box A$ for every $A \in \mathcal{L}_\Box$.

A:     $A \rhd \hat{\theta}(A)$, for every substitution $\theta$.

V(parb) :     $B \to C \rhd \bigvee_{i=1}^{n+m} \{B\}_{\mathsf{parb}}(E_i)$, in which $B = \bigwedge_{i=1}^{n}(E_i \to F_i)$
and $C = \bigvee_{i=n+1}^{n+m} E_i$, and

$$\{A\}_{\mathsf{parb}}(B) := \begin{cases} B & : B \in \mathsf{parb} \\ A \to B & : \text{otherwise} \end{cases}$$

$$\dfrac{A \rhd B \quad A \rhd C}{A \rhd B \wedge C} \text{ Conj} \qquad \dfrac{A \rhd B \quad B \rhd C}{A \rhd C} \text{ Cut}$$

$$\dfrac{B \rhd A \quad C \rhd A}{B \vee C \rhd A} \text{ Disj} \qquad \dfrac{A \rhd B \quad (C \in \mathsf{parb})}{C \to A \rhd C \to B}$$

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## Examples

$\blacktriangleleft$ back to e1

- $\lfloor x \rfloor = \bot$

- $\lfloor A \rfloor = \bot$ if $A \in \mathcal{L}_0(\text{var})$ and $A$ is not a theorem of IPC.

- $\lfloor p \rightarrow x \rfloor = \neg p$

- $\lfloor \square x \rightarrow x \rfloor = \neg \square x$

- $\lfloor \boxdot x \rightarrow y \rfloor = \neg \square x$.

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◁ iGLH $\subseteq$ iPH

## Theorem

$$[\![\text{iGL}, \text{parb}]\!]\text{LeA} = \mathop{\approx}\limits^{\text{iGL}}_{\text{SN}(\Box)} = \mathop{\approx}\limits^{\text{iGL}}_{\text{SPN}(\Box)} = \mathop{\approx}\limits^{\text{iGL}}_{\text{SPN}(\Box)^\vee}$$

*Moreover all mentioned relations are decidable.*

## Proof.

▸ Prime factorization and ▸ $\approx = \approx$ imply $\mathop{\approx}\limits^{\text{iGL}}_{\text{SN}(\Box)} = \mathop{\approx}\limits^{\text{iGL}}_{\text{SPN}(\Box)} = \mathop{\approx}\limits^{\text{iGL}}_{\text{SPN}(\Box)^\vee}$ .
▸ General soundenss implies $[\![\text{iGL}, \text{parb}]\!]\text{LeA} \subseteq \mathop{\approx}\limits^{\text{iGL}}_{\text{SN}(\Box)}$ . To show
$\mathop{\approx}\limits^{\text{iGL}}_{\text{SN}(\Box)} \subseteq [\![\text{iGL}, \text{parb}]\!]\text{LeA}$, let $A \mathop{\approx}\limits^{\text{iGL}}_{\text{SN}(\Box)} B$. Hence $\text{iGL} \vdash \lfloor A \rfloor^{\text{iGL}}_{\text{SN}(\Box)} \to B$
and thus $[\![\text{iGL}, \text{parb}]\!]\text{LeA} \vdash \lfloor A \rfloor^{\text{iGL}}_{\text{SN}(\Box)} \rhd B$. Since
$[\![\text{iGL}, \text{parb}]\!]\text{LeA} \vdash A \rhd \lfloor A \rfloor^{\text{iGL}}_{\text{SN}(\Box)}$ we are done. □

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◀ sdc

- $\mathsf{H}(\Gamma, \mathsf{T}) := \{\Box A \to \Box B : A \underset{\Gamma}{\overset{\mathsf{T}}{\approx}} B\}$.

- Hence iGLH($\Gamma$, T) is iGL plus the axiom $\mathsf{H}(\Gamma, \mathsf{T})$.

- iGLH := iGLH(C↓SN($\Box$), iGL). (provability logic of HA)

- iGLH$^\Box$ := iGLH(SN($\Box$), iGL). (complete but not sound)

- iGLC$_a$H$_\sigma$ := iGLC$_a$H(SN, iGLC$_a$).($\Sigma_1$-provability logic of HA)

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# On Provability Logic of HA

Mojtaba Mojtahedi (University of Tehran)

University of Tehran

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### Question

*Why the PLHA is not simply the following?*

$$\textsf{iGL} \textit{ plus } \square A \rightarrow \square B \textit{ for every } A \mathrel{\vnsim^{\textsf{iGL}}} B$$

- Currently the provability logic of HA is iGL plus $\square A \rightarrow \square B$ for <span style="color:red">some</span> $A, B$ satisfying $A \mathrel{\vnsim^{\textsf{iGL}}} B$. This means that $A \mathrel{\vnsim^{\textsf{iGL}}_{\textsf{C}\downarrow\textsf{SN}(\square)}} B$ implies $A \mathrel{\vnsim^{\textsf{iGL}}} B$.

- Moreover, one expects that *only* those admissible rules of iGL belong to the PLHA which HA can formalize and verify them for herself.

- $\square A \mathrel{\vnsim^{\textsf{iGL}}} A$ however HA is not able to verify it.

- Similarly following is admissible while HA can not verify: $\square(A \vee B) \rightarrow (\square A \vee \square B) \mathrel{\vnsim^{\textsf{iGL}}} \square(A \rightarrow B) \vee \square(B \rightarrow A)$.

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## $\{\!\{T, \Delta\}\!\}$ (Compare with $[\![T, \Delta]\!]$ ⊙)

$T$:     All theorems of $T$.

$V(\Delta)$:     $B \to C \rhd \bigvee_{i=1}^{n+m} \{B\}_\Delta(E_i)$, in which
$B = \bigwedge_{i=1}^{n}(E_i \to F_i)$ and $C = \bigvee_{i=n+1}^{n+m} E_i$.

$\mathrm{Mont}(\Delta)$:     $A \rhd B \to (C \to A) \rhd (C \to B)$ for every $C \in \Delta$.

$\mathrm{Le}$:     $A \rhd \Box A$ for every $A$.

$\mathrm{Disj}$:     $(B \rhd A \wedge C \rhd A) \to (B \vee C) \rhd A$.

$\mathrm{Conj}$:     $[(A \rhd B) \wedge (A \rhd C)] \to (A \rhd (B \wedge C))$.

$\mathrm{Cut}$:     $[(A \rhd B) \wedge (B \rhd C)] \to (A \rhd C)$.

$\mathrm{MP}$:     $A,\ A \to B\ /\ B$.

$\mathrm{PNec}$:     $A \to B\ /\ A \rhd B$.

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◄ PLHA

- iPH := $\{\!\!\{$iGL, parb$\}\!\!\}$. Hence
  $[\![$iGL, parb$]\!]$Le $\subseteq \{\!\!\{$iGL, parb$\}\!\!\} =$ iPH.
- Iemhoff [2003] introduced iPH and presented a proof provided by de Jongh & Visser that iPH is sound for arithmetical interpretations in HA, i.e. iPH $\vdash A$ implies HA $\vdash \alpha_{\mathsf{HA}}(A)$ for every $\alpha$. ▸
- Iemhoff [2003] conjectures that iPH is also complete for the arithmetical interpretations.
- iPH$_\sigma$ := $\{\!\!\{$iGLC$_\mathsf{a}$, atomb$\}\!\!\}$.
- The same proof implies that iPH$_\sigma$ is sound for $\Sigma_1$-interpretations in HA.
- It is quite natural to expect that iPH$_\sigma$ is also complete for $\Sigma_1$-interpretations.

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## iPH includes iGLH

◀ PLHA

### Lemma

iGLH $\vdash A$ *implies* iPH $\vdash A$.

### Proof.

By induction on the proof complexity of iGLH $\vdash A$. All cases
are trivial except for axiom instances of $\mathsf{H}(\mathsf{C}{\downarrow}\mathsf{SN}(\square), \mathsf{iGL})$,
i.e. $A = \square B \to \square C$ with $B \underset{\mathsf{C}{\downarrow}\mathsf{SN}(\square)}{\overset{\mathsf{iGL}}{\approx}} C$. ▶ implies that
$[\![\mathsf{iGL}, \mathsf{parb}]\!]\mathsf{Le} \vdash B \rhd C$ and hence iPH $\vdash B \rhd C$. ☐

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- Iemhoff [2003] provides soundness and completeness of iPH for some class of intuitionistic modal Kripke models.

$$\mathcal{K}, w \Vdash A \rhd B \quad \text{iff} \quad \forall u \sqsupset w \, (\mathcal{K}, u \Vdash A \Rightarrow \mathcal{K}, u \Vdash B).$$

- Iemhoff [2001a] also proved some partial completeness results corresponding to some fragments of iGLH.

- Mentioned Kripke models are infinite. This makes them difficult to work with.

- Here we provide Kripke-style semantic for provability and preservativity which enjoys finite-model property.

- The main idea is that we assign a proposition $\varphi_w$ to each node $w$ and

$$\mathcal{K}, w \Vdash \square B \quad \text{iff} \quad \forall u \sqsupset w \, (\, \varphi_w \vdash B)$$

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## An example

$$\nVdash \square(x \vee y) \to (\square x \vee \square y)$$



$v \Vdash \boxdot(x \vee y)$

$u \Vdash \boxdot(x \vee y)$

$\varphi_u := \boxdot x$

$\varphi_v := \boxdot y$

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## Another example

$$\nvdash \square[\square(x \vee y) \rightarrow (\square x \vee \square y)] \rightarrow \square[\square(x \rightarrow y) \vee \square(y \rightarrow x)]$$



$$\varphi_u := \boxdot[\square(x \vee y) \rightarrow (\square x \vee \square y)]$$

$$u \Vdash \square\bot$$

$$u \Vdash \boxdot[\square(x \vee y) \rightarrow (\square x \vee \square y)]$$

$$\varphi_u \vdash \square(x \vee y) \rightarrow (\square x \vee \square y)$$

$$\varphi_u \nvdash \square(x \rightarrow y) \vee \square(y \rightarrow x)$$

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### Definition

$\mathcal{K} = (W, \preccurlyeq, \sqsubset, V, \varphi)$ is called a $(\Delta, \Gamma, \mathsf{T})$-semantic if
$\tilde{\mathcal{K}} := (W, \preccurlyeq, \sqsubset, V)$ is a transitive conversely well-founded Kripke
model for the intuitionistic modal logic and

- $\varphi$ is a function on $\sqsubset$-accessible nodes of $W$ and $\varphi_w \in \Gamma$.
- $\mathcal{K}, w \Vdash \varphi_w$.
$$\mathcal{K}, w \Vdash \square A \quad \Leftrightarrow \quad \forall u \sqsupset w \; \varphi_u, \Delta_u \vdash_\mathsf{\tau} A$$

Given a set of modal propositions $Y$, define

$$Y_w := \{E \in Y : \mathcal{K}, w \Vdash E\}.$$

Whenever $\Gamma = \Delta$ we simply say that $\mathcal{K}$ is a $(\Gamma, \mathsf{T})$-semantic. In
this case it doesn't matter how $\varphi_w \in \Gamma$ are defined.

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◂ General Completeness

### Theorem

*If $A \in \Delta$-NF and $\mathcal{K}$ is $(\Delta, \Gamma, \mathsf{T})$-semantic then*

$$\tilde{\mathcal{K}}, w \Vdash A \text{ iff } \mathcal{K}, w \Vdash A.$$

### Proof.

Straightforward induction on the complexity of $A$. □

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## Theorem

*Forcing relationship for finite $(\Delta, \Gamma, \mathsf{T})$-semantic is decidable whenever $(\Delta, \mathsf{T})$ is recursively downward compact and $\mathsf{T}$ is sound.*

## Proof.

Let $\mathcal{K} = (W, \preccurlyeq, \sqsubset, V, \varphi)$ be a $(\Delta, \Gamma, \mathsf{T})$-semantic. We show decidability of $\mathcal{K}, w \Vdash A$ by double induction on $W$ ordered by $\sqsupset$ and complexity of $A$.

- $A = \square B$. It is enough to decide $\Delta_u \vdash_{\mathsf{T}} \varphi_u \to B$ for every $u \sqsupset w$. Since $(\Delta, \mathsf{T})$ is recursively downward compact, one may effectively compute $\lfloor \varphi_u \to B \rfloor_\Delta^\mathsf{T}$. By definition of $\lfloor . \rfloor_\Gamma^\mathsf{T}$ it is enough to decide $\Delta_u \vdash_{\mathsf{T}} \lfloor \varphi_u \to B \rfloor_\Delta^\mathsf{T}$ which is equivalent to $\mathcal{K}, u \Vdash \lfloor \varphi_u \to B \rfloor_\Delta^\mathsf{T}$. Then use induction hypothesis. $\quad\square$

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## Definition of Preservativity Semantic

We extend $\mathcal{K}, w \Vdash A$ to the language $\mathcal{L}_\rhd$ as follows:

$\mathcal{K}, w \Vdash B \rhd C \quad \Leftrightarrow$

$\forall u \sqsupset w \ \forall E \in \Delta \ (\Delta_u, \varphi_u \vdash_\tau E \to B \text{ implies } \Delta_u, \varphi_u \vdash_\tau E \to C),$

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## Relation to Preservativity

◀ Soundness

### Theorem

$\mathrel{\underset{\Gamma}{\overset{\tau}{\approx}}}$ is sound for $(\Delta, \Gamma, \mathsf{T})$-semantics, i.e. given such preservativity semantics $\mathcal{K}$, we have $\mathcal{K} \Vdash A \rhd B$ whenever $A \mathrel{\underset{\Gamma}{\overset{\tau}{\approx}}} B$.

### Proof.

Let $A \mathrel{\underset{\Gamma}{\overset{\tau}{\approx}}} B$ and $\mathcal{K} = (W, \preccurlyeq, \sqsubset, V, \varphi)$ be a $(\Delta, \Gamma, \mathsf{T})$-semantics and $w \sqsubset u \in W$ and $E \in \Delta$ such that $\varphi_u, \Delta_u, E \vdash_\tau A$. Hence there is a finite set $\Phi_u \subseteq \Delta_u$ such that $\Phi_u, E, \varphi_u \vdash A$. By conjunctive closure condition, we have $\bigwedge \Phi_u \wedge E \wedge \varphi_u \in \Gamma$ and thus by $A \mathrel{\underset{\Gamma}{\overset{\tau}{\approx}}} B$ we get $\Phi_u, E, \varphi_u \vdash_\tau B$. Hence we have $\varphi_u, \Delta_u, E \vdash_\tau B$. $\qquad\square$

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## Soundness

### Theorem

iGLH($\Gamma$, T) *is sound for* $(\Delta, \Gamma, \mathsf{T})$-*semantics whenever* IPC $\subseteq$ T *and* $\mathsf{SN}(\square) \subseteq \Delta$.

- The proof is by induction on $A \in \mathcal{L}_\square$ and $W$ ordered by $\sqsupset$.
- One may use ▶ to show soundness of H($\Gamma$, T).
- $\mathsf{SN}(\square) \subseteq \Delta$ is needed for soundness of iGL and necessitation.
- The proof is straightforward.

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## Completeness

### Theorem

iGLH$(\Gamma, \mathsf{T})$ *is complete for good* $(\Gamma, \mathsf{T})$*-semantics, if* $(\Gamma, \mathsf{T})$ *is sdc and* $\Gamma \supseteq \mathsf{SN}(\square)$ *is closed under conjunctions and* $\mathsf{T} \supseteq \mathsf{IPC}$.

### Proof.

Let iGLH$(\Gamma, \mathsf{T}) \nvdash A$. Then by ▶ we also have
iGLH$(\Gamma, \mathsf{T}) \nvdash \lVert A \rVert_\Gamma^\mathsf{T}$ and a fortiori iGL $\nvdash \lVert A \rVert_\Gamma^\mathsf{T}$. ▶ implies that
there is some good Kripke model $\tilde{\mathcal{K}} := (W, \preccurlyeq, \sqsubset, V)$ such that
$\tilde{\mathcal{K}}, w_0 \nVdash \lVert A \rVert_\Gamma^\mathsf{T}$. Define $\mathcal{K} := (W, \preccurlyeq, \sqsubset, V, \varphi)$ and choose $\varphi_w \in \Gamma$
as you want. ▶ implies that $\lVert A \rVert_\Gamma^\mathsf{T} \in \Gamma\text{-}\mathsf{NF}$ and ▶ implies
$\mathcal{K} \nVdash \lVert A \rVert_\Gamma^\mathsf{T}$. Then soundness implies desired result. $\square$

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◄ Second reduction

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### Corollary

iGLC$_a$H$_\sigma$ *is sound and complete for* C$_a$ *good* (SN, iGLC$_a$)*-semantics.*

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### Corollary

iGLH$^\square$ *is sound and complete for good* (SN($\square$), iGL)*-semantics.*

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### Corollary

iGLH *is sound for* (SN($\square$), C↓SN($\square$), iGL)*-semantics.*

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◀ 1st-red

- Since $C{\downarrow}SN(\square)$ is not closed under conjunctions we do not have completeness of iGLH for $(C{\downarrow}SN(\square), iGL)$-semantics.

- Like most useful results, the proof of following theorem is not easy!

- Its proof needs its own saturation and truth lemmas.
- See the manuscript for details.

### Theorem

iGLH *is complete for good* $(SN(\square), C{\downarrow}SN(\square), iGL)$-*semantics.*

### Corollary

iGLH *is decidable.*

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# Back to example $\nvdash \square(x \vee y) \rightarrow (\square x \vee \square y)$

- That model ▶ is $(\mathsf{SN}(\square), \mathsf{C}{\downarrow}\mathsf{SN}(\square), \mathsf{iGL})$-semantic.
- $\mathcal{K}, w \nVdash \square y$:

### Proof.

Enough to show $\mathsf{SN}(\square)_u, \boxdot x \nvdash y$. If $\mathsf{SN}(\square)_u \vdash \boxdot x \rightarrow y$, then
$\mathsf{SN}(\square)_u \vdash \lfloor \boxdot x \rightarrow y \rfloor_{\mathsf{SN}(\square)}^{\mathsf{iGL}}$. As we saw earlier ▶,
$\lfloor \boxdot x \rightarrow y \rfloor_{\mathsf{SN}(\square)}^{\mathsf{iGL}} = \neg\square x$. Thus $\mathsf{SN}(\square)_u \vdash \neg\square x$. Hence by
soundness of $\mathsf{iGL}$ we have $\mathcal{K}, u \Vdash \neg\square x$, a contradiction. ❑

- This shows that $\mathsf{iGLH} \nvdash \square(x \vee y) \rightarrow (\square x \vee \square y)$.

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### Theorem

*The provability logic of* HA *is* iGLH *and hence is decidable.*

### Proof.

Soundness: iGLH $\vdash A$ implies HA $\vdash \alpha_{\mathsf{HA}}(A)$ for every $\alpha$.
⚫ implies iGLH $\subseteq$ iPH. Then by soundness of iPH ⚫ we have
desired result.

Completeness: iGLH $\nvdash A$ implies HA $\nvdash \alpha_{\mathsf{HA}}(A)$ for some $\alpha$.
iGLH $\nvdash A$ implies iGLC$_{\mathsf{a}}$H$_\sigma \nvdash \theta(A)$ for some propositional modal
substitution $\theta$ (we will see later). Then arithmetical
completeness of iGLC$_{\mathsf{a}}$H$_\sigma$ implies desired result. ❑

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# What happens without $\mathsf{H}$ and $\mathsf{H}_\sigma$

◀ 2nd-red

### Theorem

$\mathsf{iGL} \nvdash A$ implies $\mathsf{iGLC_a} \nvdash \beta(A)$ for some propositional substitution $\beta$.

### Proof.

Since $\mathsf{iGL} \nvdash A$, there is some $\mathcal{K} := (W, \preccurlyeq, \sqsubset, V)$ with $\mathcal{K} \nVdash A$ ▶.
Define $\mathcal{K}'$: for every $w \in W$ add a fresh atomic $p_w$ and let it be forced (satisfied) at $w$ and its successor/above nodes. No other atomics are forced at $w$.
Define $\beta(x) := \bigvee_{\mathcal{K}, w \Vdash x} Q_w$ and $Q_w := q_w \wedge \bigwedge_{w \sqsubset u} \neg q_u$.
Claim. $\mathcal{K}, w \Vdash A$ iff $\mathcal{K}', w \Vdash \beta(A)$. ◻

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## iGLH $\nvdash A$ implies iGLC$_a$H$_\sigma$ $\nvdash \theta(A)$

The proof is broken in two steps:

1. iGLH $\nvdash A$ implies iGLH$^\square$ $\nvdash \gamma(A)$ for some $\gamma$.

2. iGLH$^\square$ $\nvdash A$ implies iGLC$_a$H$_\sigma$ $\nvdash \beta(A)$ for some $\beta$.

Both are proved via provability semantics.

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# $\mathsf{iGLH}^\square \nvdash A$ implies $\mathsf{iGLC_aH}_\sigma \nvdash \beta(A)$

### Sketch of the proof.

- Since $\mathsf{iGLH}^\square \nvdash A$, by completeness for $(\mathsf{SN}(\square), \mathsf{iGL})$-semantics ▶, $\mathcal{K} \nVdash A$ for some $\mathcal{K} = (W, \preccurlyeq, \sqsubset, V, \top)$.

- On the other hand, $\mathsf{iGLC_aH}_\sigma$ is sound for $\mathsf{CP_a}$ $(\mathsf{SN}(\square), \mathsf{SN}, \mathsf{iGLC_a})$-semantics. ▶

- One must transform $\mathcal{K}$ to a $(\mathsf{SN}(\square), \mathsf{SN}, \mathsf{iGLC_a})$-semantic.

- The transformation is a uniform collection of transformations for $\mathsf{iGL}$ and $\mathsf{iGLC_a}$ ▶.

- See the manuscript for details.

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# iGLH $\nvdash A$ implies iGLH$^\Box$ $\nvdash \gamma(A)$

- The first step reduction is not as elementary as the second one.

- It uses features of relative projectivity and simultaneous fixed point theorem.

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# iGLH $\nvdash A$ implies iGLH$^\Box \nvdash \gamma(A)$

### Sketch of the proof.

- iGLH $\nvdash A$ implies $\mathcal{K} \nVdash A$ for some good $(\mathsf{SN}(\Box), \mathsf{C}{\downarrow}\mathsf{SN}(\Box), \mathsf{iGL})$-semantic $\mathcal{K} = (W, \preccurlyeq, \sqsubset, V, \varphi)$ ▶.

- Since $\varphi_w \in \mathsf{C}{\downarrow}\mathsf{SN}(\Box)$ there is an $(\mathsf{SN}(\Box), \mathsf{iGL})$-projective substitution $\theta_w$ such that $\hat{\theta}_w(\varphi_w) \in \mathsf{SN}(\Box)$.

- The main idea is that one-bye-one we must kill $\varphi_w$'s and send them in to the set $\mathsf{SN}(\Box)$.

- Good news: when some $\varphi_w$ goes in to $\mathsf{SN}(\Box)$ it remains there since $\mathsf{SN}(\Box)$ is closed under substitutions.

- Bad news: These $\hat{\theta}_w$'s are not even substitutions.

- Solution: simultaneous fixed point theorem in iGL ▶. □

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# Thanks For Your Attention

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