On provability logic of Heyting Arithmetic

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Overview

Solovay 1976

The Provability logic of PA is GL

- All theorems of classical propositional logic.
- $\mathsf{K} := \Box(A \to B) \to (\Box A \to \Box B)$.
- $\mathsf{Löb} := \Box(\Box A \to A) \to \Box A$. Implies $\Box A \to \Box\Box A$.
- modus ponens: $A, A \to B / B$.
- Necessitation: $A / \Box A$.

Axiomatization: roughly

iGL plus $\Box A \to \Box B$ for every $A \mathrel{|\!\!\sim} B$

Provability logic of HA

- A. Visser 1980 first considered this.
- Since then many partial related results where obtained. We will see some of them later.
- Main source for difficulty: HA-verifiable admissible rules.

$$\frac{\neg A \to (B \vee C)}{(\neg A \to B) \vee (\neg A \to C)}$$

Admissible rules

- $A \mathrel{|\!\!\sim_{\mathsf{T}}} B$ iff $\forall \alpha \ (\mathsf{T} \vdash \alpha(A) \Rightarrow \mathsf{T} \vdash \alpha(B))$.
- Example: $\neg A \to (B \vee C) \mathrel{|\!\!\sim_{\mathsf{IPC}}} (\neg A \to B) \vee (\neg A \to C)$.
- In the provability logic of HA, the above rule reflected as:

$$\Box(\neg A \to (B \vee C)) \to \Box((\neg A \to B) \vee (\neg A \to C)).$$

- Why not classically interesting?

$$A \mathrel{|\!\!\sim_{\mathsf{CPC}}} B \quad \text{iff} \quad \mathsf{CPC} \vdash A \to B.$$

Admissible rules of IPC

- For every $A \mathrel{|\!\!\sim_{\mathsf{IPC}}} B$ we have $\Box A \to \Box B$ in $\mathsf{PL}(\mathsf{HA})$.
- What are the admissible rules of IPC? Decidable? (H. Friedman 1975)

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  (H. Friedman 1975)
  Decidability: Rybakov 1997.
  Axiomatization: Visser and de Jongh (??).
  Completeness proof: Iemhoff 2001.

The system $[\![\mathsf{T}, \Delta]\!]$

Axioms: Define $A \xrightarrow{\Delta} E := \begin{cases} E & : E \in \Delta \\ A \to E & : \text{otherwise} \end{cases}$

$$\frac{\mathsf{T} \vdash A \to B}{A \rhd B}\ [\mathsf{T}]$$

$$\frac{A = \bigwedge_{i=1}^{n}(E_i \to F_i) \qquad B = \bigvee_{i=n+1}^{n+m}(F_i)}{(A \to B) \rhd \bigvee_{i=1}^{n+m} A \xrightarrow{\Delta} E_i}\ \mathsf{V}(\Delta)$$

Rules:

$$\frac{A \rhd B \qquad A \rhd C}{A \rhd (B \wedge C)}\ \text{Conj} \qquad \frac{A \rhd B \qquad B \rhd C}{A \rhd C}\ \text{Cut}$$

$$\frac{A \rhd C \qquad B \rhd C}{(A \vee B) \rhd C}\ \text{Disj} \qquad \frac{A \rhd B \qquad (D \in \Delta)}{(D \to A) \rhd (D \to B)}\ \text{Mont}(\Delta)$$

Admissible Rules of IPC

Theorem (Iemhoff 2001)

$A \mathrel{|\!\!\sim}_{\mathsf{IPC}} B$ *iff* $[\![\mathsf{IPC}, \mathsf{cons}]\!] \vdash A \rhd B$.

Theorem (Visser 2002)

$A \mathrel{|\!\!\sim}_{\mathsf{IPC}} B$ *iff* $[\![\mathsf{IPC}, \{\top, \bot\}]\!] \vdash A \rhd B$ *iff* $\Box A \to \Box B \in \mathsf{PL}(\mathsf{HA})$.

What else in PL(HA)? (Disjunction property)

- DP means that if a disjunction is derivable, then either of them are derivable.
- IPC, IQC and HA has DP.
- $\mathsf{CPC} \vdash p \vee \neg p$ while $\mathsf{CPC} \nvdash p$ and $\mathsf{CPC} \nvdash \neg p$.
- $\Box(A \vee B) \to (\Box A \vee \Box B) \in \mathsf{PL}(\mathsf{HA})$?
- H. Friedman 1975: No!
- D. Leivant 1975: $\Box(A \vee B) \to \Box(\boxdot A \vee \boxdot B) \in \mathsf{PL}(\mathsf{HA})$.
- Above axiom together with reflection implies DP.

What else in PL(HA)? (Markov Rule)

$$\forall S \in \Sigma_1 \ \big(\mathsf{HA} \vdash \neg\neg S \quad \text{implies} \quad \mathsf{HA} \vdash S\big).$$

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Theorem (Visser 1981)

$$\Box\neg\neg\Box A \to \Box\Box A \in \mathsf{PL}(\mathsf{HA}).$$

Theorem (Visser 1981)

The letterless fragment of PL(HA) *is decidable.*

PL(HA): the axiomatization

Let us define the Leivant's axiom schema as follows:

$$(\mathsf{Le}):\ A \rhd \Box A \text{ for every } A.$$

Theorem (M. 2022)

$\mathsf{iGLH} := \mathsf{iGL} + \{\Box A \to \Box B : [\![\mathsf{iGL}, \Box]\!] + \mathsf{Le} \vdash A \rhd B\} = \mathsf{PL(HA)}.$

Arithmetical soundness

The arithmetical soundness of this system in a more general setting, namely Σ_1-preservativity, was already known by Visser, de Jongh and Iemhoff (2001).

Arithmetical Completeness: very roughly

Propositional reduction to the Σ_1-provability logic.

Reduction: Classical

Theorem (Visser)

$\mathsf{PL}_\Sigma(\mathsf{PA}) = \mathsf{GLC_a} := \mathsf{GL} + p \to \Box p$ *for atomic* p*'s.*

Theorem (Ardeshir & M. 2015)

One may reduce the arithmetical completeness of GL *to the one for* $\mathsf{GLC_a}$*.*

Reduction example

$$A := p \to (\Box p \vee \Box \neg p) \qquad \mathsf{GL} \nvdash A$$

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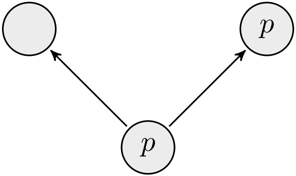

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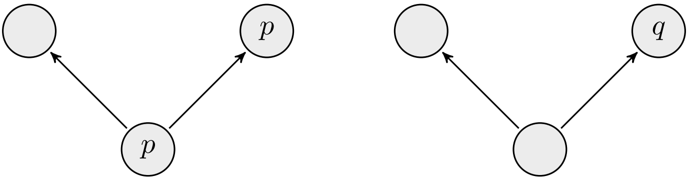

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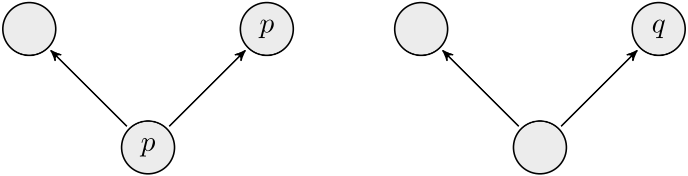

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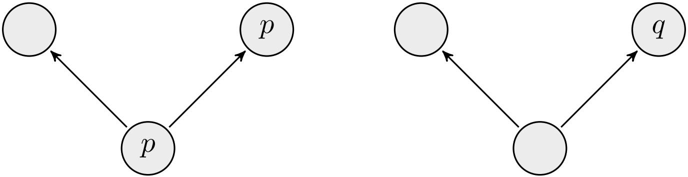

$$\tau(p) := q \vee \neg \Box \bot \quad \Rightarrow \quad \mathsf{GLC_a} \nvdash \tau(A) \quad \Rightarrow \quad \mathsf{PA} \nvdash \sigma\tau(A)$$

Σ_1-provability logic of HA

Theorem (Ardeshir & M. 2018)

$\mathsf{iGLC_aH_\sigma} = \mathsf{PL}_\Sigma(\mathsf{HA})$ *in which* $\mathsf{iGLC_aH_\sigma} :=$
$\mathsf{iGLC_a} + \{\Box A \to \Box B : [\![\mathsf{iGLC_a}, \Box \cup \mathsf{atomic}]\!] + \mathsf{Le} \vdash A \rhd B\}$

Arithmetical Completeness of iGLH

1. Let $\mathsf{iGLH} \nvdash A$.
2. find some α s.t. $\mathsf{iGLC_aH_\sigma} \nvdash \alpha(A)$.
3. use arithmetical completeness of $\mathsf{iGLC_aH_\sigma}$ to find σ s.t. $\mathsf{HA} \nvdash \sigma\alpha(A)$.

Step 2

- We first need a finite, or at least well-behaved Kripke semantics.
- Iemhoff already provided a semantic for an extension of iGLH in the language with binary modal operator.
- Iemhoff's semantics are not finite.
- At least we failed to use it for the purpose of reduction.
- We provided a finite *mixed* semantic which is a combination of derivability and Kripke-style validity.
- The mixed semantic is sewed for preservativity.

Preservativity

$A \mathrel{|\!\!\!\approx^{\Gamma}_{\mathsf{T}}} B$ iff for every $E \in \Gamma$ ($\mathsf{T} \vdash E \to A$ implies $\mathsf{T} \vdash E \to B$)

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Observation.

$A \mathrel{|\!\!\sim_{\mathsf{IPC}}} B$ iff $A \mathrel{|\!\!\approx^{\Gamma}_{\mathsf{IPC}}} B$.

Γ is the set of projective propositions:

$$\exists\, \theta,\ \vdash \theta(A) \quad \text{and} \quad A \vdash \theta(x) \leftrightarrow x$$

Inspired by algebraic notion of projectivity, Silvio Ghilardi introduced this notion.

Theorem (Unification in IL: Ghilardi 1999)

For every A, there is a finite set Π_A of projective formulas s.t.

- $\vdash \bigvee \Pi_A \to A$.
- $A \mathrel{|\!\!\sim_{\mathsf{IPC}}} \bigvee \Pi_A$.

Proof of the observation

- Let $A \mathrel{|\!\!\approx^{\Gamma}_{\mathsf{IPC}}} B$ and $\vdash \theta(A)$.
- By Ghilardi's theorem, there is $E \in \Pi_A$ s.t. $\vdash \theta(E)$.
- $A \mathrel{|\!\!\approx^{\Gamma}_{\mathsf{IPC}}} B$ and $\vdash E \to A$ implies $\vdash E \to B$.
- Thus $\vdash \theta(B)$.

For other way around:

- Let $A \mathrel{|\!\!\sim_{\mathsf{IPC}}} B$ and projective E s.t. $\vdash E \to A$.
- By projectivity of E we have $\vdash \theta(A)$.
- $A \mathrel{|\!\!\sim_{\mathsf{IPC}}} B$ implies $\vdash \theta(B)$.
- $\vdash E \to \theta(B)$.
- $\vdash E \to B$.

Parametric Unification

- Extend the language with atomic parameters.
- Substitutions does not replace parameters. (identity on par)
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- Example: $x \to p$. $(\tau(x) := x \wedge p$ and $\tau(p) = p)$
- Example: $x \wedge p$. Not unifiable! Thus not projective.
- Instead of *unification* we try to *simplify* a given formula to $\mathsf{Bool}(\mathsf{par})$.
- Example: $x \wedge p$. $(\tau(x) := \top$ and $\tau(p) := p)$ This τ is projective and it projects to p.

NNIL(par)-fication

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$$A \mathrel{|\!\!\sim^{\Gamma}_{\mathsf{T}}} B \quad \Leftrightarrow \quad \forall E \in \Gamma \, \forall \theta, \; \mathsf{T} \vdash \theta(E \to A) \Rightarrow \mathsf{T} \vdash \theta(E \to B)$$

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$$A \mathrel{|\!\sim^{\Gamma}_{\mathsf{T}}} B \quad \Leftrightarrow \quad \forall\, E \in \Gamma\, \forall\, \theta,\ \mathsf{T} \vdash \theta(E \to A) \Rightarrow \mathsf{T} \vdash \theta(E \to B)$$

A is NNIL(par)-projective if there is some θ and $B \in \mathsf{NNIL}(\mathsf{par})$ s.t. $\mathsf{IPC} \vdash \theta(A) \leftrightarrow B$ and $A \vdash_{\mathsf{IPC}} \theta(x) \leftrightarrow x$ for every var x.

NNIL(par)-projectivity

Theorem (M. 2022)

For every A, there is a finite set Π_A of NNIL(par)*-projective formulas s.t.*

- $\vdash \bigvee \Pi_A \to A$.
- $A \mathrel{|\!\!\sim}^{\mathsf{N(par)}}_{\mathsf{IPC}} \bigvee \Pi_A$.

Theorem (M. 2022)

$A \mathrel{|\!\!\sim}^{\mathsf{N(par)}}_{\mathsf{IPC}} B$ *iff* $[\![\mathsf{IPC}, \mathsf{NNIL(par)}]\!] \vdash A \rhd B$ *iff* $A \mathrel{|\!\!\approx}^{\downarrow\mathsf{N(par)}}_{\mathsf{IPC}} B$.

With the aid of parametric unification and admissibility, and also the fixed point lemma in iGL we may prove:

Theorem (M. 2022)

$[\![\mathsf{iGL}, \Box]\!] + \mathsf{Le} \vdash A \rhd B$ *iff* $A \mid\!\approx^{\Gamma}_{\mathsf{iGL}} B$.

$\Gamma := \mathsf{C{\downarrow}SN}(\Box)$
Roughly, Γ is the set of modal propositions which could be projected to a $\mathsf{NNIL}(\Box)$-proposition.

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Roughly, Γ is the set of modal propositions which could be projected to a $\mathsf{NNIL}(\Box)$-proposition.

We first consider the non-modal version of this statement, with parameters as boxed formulas.

Mixed semantic

- Roughly speaking, a mixed semantic is a usual Kripke model for intuitionistic modal logic, with the following strengthen for validity of $\Box A$ at w:

$$w \Vdash \Box A \quad \text{iff} \quad \forall u \sqsupset w$$

a fragment of valid propositions at u implies A

- It is a mixture of validity and derivability.

$\mathsf{iGLH}(\Gamma, \mathsf{T})$

$$\mathsf{iGLH}(\Gamma, \mathsf{T}) := \mathsf{iGL} + \{\Box A \to \Box B : A \mathrel{\substack{\Gamma \\ \approx \\ \mathsf{T}}} B\}$$

Mixed semantic for $\mathsf{iGLH}(\Gamma, \mathsf{T})$

- Δ: a set of propositions such that both Δ and Γ are closed under Δ-conjunctions.
- $\varphi_w \in \Gamma$,
- $\mathcal{K}, w \Vdash \phi_w$,
- $\Delta_w := \{A \in \Delta : w \Vdash A\}$,
- $\mathcal{K}, w \Vdash \Box A$ iff for every $u \sqsupset w$ we have $\mathsf{T}, \Delta_w, \varphi_u \vdash A$.

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Theorem (Soundness)

Mixed semantics are sound for $\mathsf{iGLH}(\Gamma, \mathsf{T})$.

Theorem (Completeness I (M. 2022))

($\Delta := \Box$ and Γ is $\mathsf{NNIL}(\Box)$-projectives) Finite mixed semantics are complete for $\mathsf{iGLH}(\Gamma, \mathsf{iGL})$.

Theorem (Completeness II (M .2022))

($\Delta = \Gamma = \mathsf{NNIL}(\Box)$ and $\varphi_w = \top$) Finite mixed semantics are complete for $\mathsf{iGLC_aH}(\Gamma, \mathsf{iGLC_a})$.

Then we transform Kripke models of first type to the models of second type.

Future Works: Relative Provability Logics

- $\mathsf{PL}(\mathsf{T}, \mathsf{S})$.

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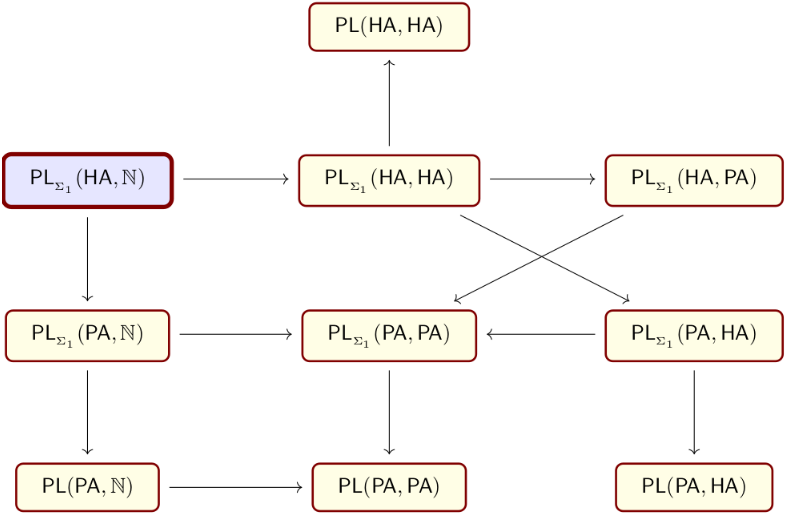
- $\mathsf{PL}(\mathsf{T}, \mathsf{S})$.
- The classical version have been characterized by series of publications by Artemov, Visser, Beklemishev and Japaridze. (1980-1989)
- We have the same question for the intuitionistic case.
- Many relative provability logics are already known.

Known results



Parametric Unification and admissibility for classical modal logics.

Intuitionistic variant of Interpretability Logic, is the Σ_1-preservativity. Its arithmetical completeness is still open.

Thanks For Your Attention