Reduction of arithmetical completenesses

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Provability Logic

- $\Box$ interpreted as provability.

Provability Logic: more precise

$$\mathsf{PL}(T) := \text{Provability logic of } T := \{A \in \mathcal{L}_\Box : \forall \sigma \; T \vdash \sigma_{_T} A\}$$

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- $\sigma_{_T}(\Box A) := \mathsf{Pr}_{_T}(\ulcorner \sigma_{_T} A \urcorner)$.

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The Provability logic of PA is GL

- All theorems of classical propositional logic.
- K $:= \Box(A \to B) \to (\Box A \to \Box B)$.
- Löb $:= \Box(\Box A \to A) \to \Box A$.
  Implies $\Box A \to \Box\Box A$. (Due to Dick de Jongh)
- modus ponens: $A, A \to B / B$.
- Necessitation: $A/\Box A$.

Σ_1-provability logics

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The Σ_1-Provability logic of PA is $\mathsf{GLC_a} := \mathsf{GL} + p \to \Box p$

Ardeshir & M. 2015

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- (Assuming the arithmetical completeness of $\mathsf{GLC_a}$) $\mathsf{GLC_a} \nvdash \tau A$ implies $\mathsf{PA} \nvdash \sigma_{\mathsf{PA}} \tau A$ for some Σ_1-sub.
- $\mathsf{PA} \nvdash (\sigma\tau)_{\mathsf{PA}}(A)$.

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Arithmetical completeness of $\mathsf{PL}(\mathsf{PA})$ propositionally reducible to the one for $\mathsf{PL}_\Sigma(\mathsf{PA})$.

Relative Provability Logic

$$\mathsf{PL}_\Gamma(T, U) := \Gamma\text{-Provability logic of } T \text{ relative in } U$$
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Another example

Theorem (Solovay 1976)

$\mathsf{PL}(\mathsf{PA}, \mathbb{N}) = \mathsf{GLS} := \mathsf{GL} + \Box A \to A.$

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Theorem (M. 2021)

$\mathsf{PL}(\mathsf{PA}, \mathsf{PA})$ *is reducible to* $\mathsf{PL}(\mathsf{PA}, \mathbb{N})$.

Proof. Let $\mathsf{GL} \nvdash A$. Then $\mathsf{GL} \nvdash \Box A$ and hence $\mathsf{GLS} \nvdash \Box A$. Then $\mathbb{N} \not\models \alpha_{\mathsf{PA}}(\Box A)$. Thus $\mathsf{PA} \nvdash \alpha_{\mathsf{PA}}(A)$.

Yet another example

Theorem (M. 2022)

$\mathsf{PL}(\mathsf{HA},\mathsf{HA})$ *is decidable and is as follows:*

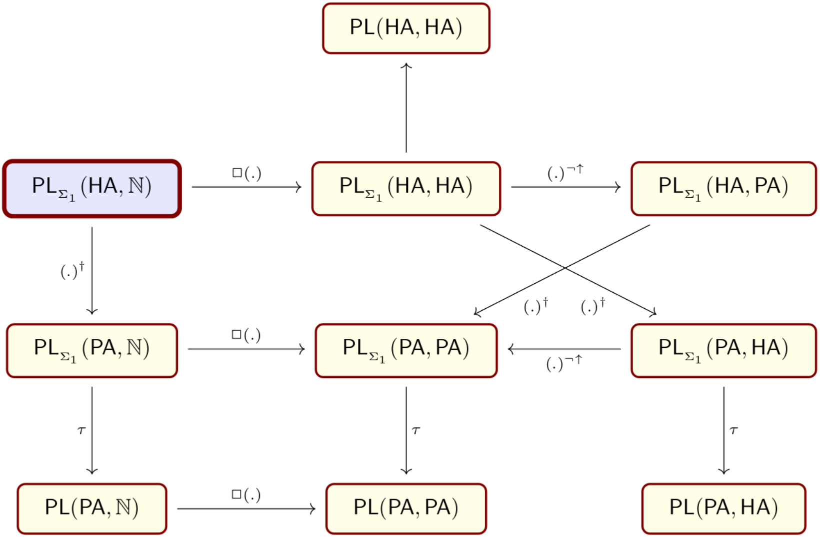
$$\mathsf{iGLH} := \mathsf{iGL} + \{\Box A \to \Box B : A \mathrel{|\!\sim} B\}$$

The proof includes just the reduction of $\mathsf{PL}(\mathsf{HA},\mathsf{HA})$ to $\mathsf{PL}_{\Sigma}(\mathsf{HA},\mathsf{HA})$.

Theorem (M. 2018)

$\mathsf{PL}_{\Sigma}(\mathsf{HA},\mathsf{HA})$ *is decidable and is as follows:*

$$\mathsf{iGLH}_{\sigma} := \mathsf{iGLC_a} + \{\Box A \to \Box B : A \mathrel{|\!\sim}_{\Sigma_1} B\}$$



PL(HA, HA)
PL$_{\Sigma_1}$(HA, ℕ)
□(.)
PL$_{\Sigma_1}$(HA, HA)
(.)$^{\neg\uparrow}$
PL$_{\Sigma_1}$(HA, PA)
(.)†
(.)†
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τ
τ
τ
PL(PA, ℕ)
□(.)
PL(PA, PA)
PL(PA, HA)

Negative result

Theorem (F. Pakhomov)

$\mathsf{PL}_\Sigma(\mathsf{PA}, \mathsf{PA})$ *is not reducible to* $\mathsf{PL}(\mathsf{PA}, \mathsf{PA})$.

General definition

Definition

We say that f, F propositionally reduces $\mathcal{AC}_{\Gamma'}(V'; T', U')$ to $\mathcal{AC}_{\Gamma}(V; T, U)$, with the notation $\mathcal{AC}_{\Gamma}(V; T, U) \leq^{\text{Prop}}_{f,F} \mathcal{AC}_{\Gamma'}(V'; T', U')$, if:

1. $f : \mathcal{L}_\Box \longrightarrow \mathcal{L}_\Box$ and $F = \{F_A : A \in \mathcal{L}_\Box\}$ is a family of functions,
2. $V \vdash f(A)$ implies $V' \vdash A$,
3. F_A is a function on arithmetical substitutions and

$$F_A : [\![f(A); T, U; \Gamma]\!] \longrightarrow [\![A; T', U'; \Gamma']\!] \text{ and } F_A(\sigma) \in [\sigma, T'].$$

$[\sigma, T]$ is propositional closure of σ, the smallest set X s.t. (1) $\sigma \in [\sigma, T]$ and (2) $\alpha \in [\sigma, T]$ implies $\alpha_{\top}\tau \in [\sigma, T]$.

Future works

1. Are all mentioned reductions strict?
2. All *classical* provability logics (classified by ABGV).
3. What about other arithmetical completenesses: Interpretability logic and GLP.

Thanks For Your Attention