Polymodal Σ-Provability Logic

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Provability Logic

- $\Box$ interpreted as provability.
- Gödel 1933: Based on BHK.

Provability Logic: more precise

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- $\sigma_T(p) := \sigma(p)$ for atomics.
- σ_T commutes with boolean connectives.
- $\sigma_T(\Box A) := \mathsf{Pr}_T(\ulcorner \sigma_T A \urcorner)$.

The Provability logic of PA is GL

- All theorems of classical propositional logic.
- $\mathsf{K} := \Box(A \to B) \to (\Box A \to \Box B)$.
- $\mathsf{L\ddot{o}b} := \Box(\Box A \to A) \to \Box A$.
  Implies $\Box A \to \Box\Box A$. (Due to Dick de Jongh)
- modus ponens: $A, A \to B / B$.
- Necessitation: $A / \Box A$.

The Σ_1-provability logic

$$\mathsf{PL}_{\Sigma}(T) := \Sigma_1\text{-Provability logic of } T := \{A \in \mathcal{L}_{\Box} : \forall \sigma\ T \vdash \sigma_T A\}$$

- $\sigma_T(p) := \sigma(p) \in \Sigma_1$ for atomics.
- σ_T commutes with boolean connectives.
- $\sigma_T(\Box A) := \mathsf{Pr}_T(\ulcorner \sigma_T A \urcorner)$.

Visser 1980

The Σ_1-Provability logic of PA is $\mathsf{GL} + \mathsf{C_a}$

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- modus ponens: $A, A \to B / B$.
- Necessitation: $A / \Box A$.
- $\mathsf{C_a} := p \to \Box p$ for atomic p.

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- **Reducible:** Only via propositional translations.
- $\mathsf{GL} \nvdash A$ implies $\mathsf{GL} + \mathsf{C_a} \nvdash \tau(A)$ for some τ. Then by A.C. of $\mathsf{GL} + \mathsf{C_a}$, there is σ s.t. $\mathsf{PA} \nvdash \sigma_{\mathsf{PA}}(\tau(A))$. Take $\alpha := \sigma \circ \tau$ and then $\mathsf{PA} \nvdash \alpha_{\mathsf{PA}}(A)$.

Ardeshir, M., & Mojtahedi, M. (2015). Reduction of provability logics to Σ_1-provability logics. Logic Journal of the IGPL, 23(5), 842-847.

- A precise definition for *reduction* of provability logics.
- The Σ_1-provability logic of HA relative in the standard model is the hardest.

Mojtahedi, M. (2021). Hard provability logics. Mathematics, Logic, and their Philosophies: Essays in Honour of Mohammad Ardeshir, 253-312.

F. Pakhomov 2023

Arithmetical completeness of $\mathsf{PL}_\Sigma(\mathsf{PA})$ is not reducible to that of $\mathsf{PL}(\mathsf{PA})$.

Reduction of PL(HA) to PL_{Σ}(HA)

Theorem (M. 2022)

PL(HA) *is reducible to* PL_{Σ}(HA).

Mojtahedi, M. (2022). On provability logic of HA. arXiv preprint arXiv:2206.00445.

Ardeshir, M., & Mojtahedi, M. (2018). The Σ_1-provability logic of HA. Annals of Pure and Applied Logic, 169(10), 997-1043.

$\mathsf{PL}(\mathsf{PA}^n, \mathsf{PA})$

- PA^n is PA plus all true Π^0_n-sentences.
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$$\mathsf{PL}(\mathsf{PA}^n, \mathsf{PA}) := \text{Provability logic of } \mathsf{PA}^n \text{ in } \mathsf{PA}$$
$$:= \{A \in \mathcal{L}_\Box : \forall\sigma\ \mathsf{PA} \vdash \sigma_{\mathsf{PA}^n} A\}$$

- $\mathsf{PL}(\mathsf{PA}^n, \mathsf{PA}) := \mathsf{GL}$.

All PA^n's together

- What happens if we consider all PA^n's provability predicates at once?
- We extend first the propositional language.
- We add a unary modal operator $[n]$ for every $n \in \omega$. ($\mathcal{L}_\omega$)
- What are all arithmetically valid $A \in \mathcal{L}_\omega$?

GLP: Japaridze 1986

The system GLP axiomatized over $\mathcal{L}_\omega$:

- All axioms of Classical Logic
- $[n](A \to B) \to ([n]A \to [B])$.
- $[n]([n]A \to A) \to [n]A$.
- $[n]A \to [n+1]A$.
- $\neg[n]A \to [n+1]\neg[n]A$.
- $(A, A \to B)/B$.
- $A/[0]A$.

Theorem (Japaradize)

GLP $\vdash A$ *iff* PA $\vdash \alpha_\omega(A)$ *for every* α.

GLP is . . .

- interesting due to its applications in the study of proof-theoretic strength of theories.
- not sound and complete for any finite Kripke frames.
- sound and complete for Topological semantics.

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Theorem (M. 2024)

The polymodal Σ_n-provability logic of PA *is* GLP_n.

$$\mathsf{GLP}_n := \mathsf{GLP} \quad + \quad p \to [n]p \quad + \quad \neg p \to [n+1]p$$

Reduction to polymodal Σ_n-provability logic

Theorem (M. 2024)

Arithmetical completeness of GLP *is reducible to* $\{\mathsf{GLP}_n\}_{n\in\omega}$.

This means that from $\mathsf{GLP} \nvdash A$, we can find some *propositional substitution* τ and some $n \in \omega$ s.t. $\mathsf{GLP}_n \nvdash \tau(A)$.

Future Works

The intuitionistic version of poly-modal provability. We know (in a joint work with F. Pakhomov) that HA^n, extension of HA by all true Π^0_n-sentences, has the same provability logic as HA.

Is GLP_ω strictly harder than GLP?

Exploring more reductions for polymodal provability logics.

Thanks For Your Attention