A translation from GLP to GL_ω

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Provability Logic

- $\Box$ interpreted as provability.
- Gödel 1933: Based on BHK.

Provability Logic: more precise

$$\mathsf{PL}(T) := \text{Provability logic of } T := \{A \in \mathcal{L}_\Box : \forall \sigma \; T \vdash \sigma_{_T} A\}$$

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- $\sigma_T(p) := \sigma(p)$ for atomics.
- σ_T commutes with boolean connectives.
- $\sigma_T(\Box A) := \mathsf{Pr}_T(\ulcorner \sigma_T A \urcorner)$.

Solovay 1976

The Provability logic of PA is GL

- All theorems of classical propositional logic.
- $\mathsf{K} := \Box(A \to B) \to (\Box A \to \Box B)$.
- $\mathsf{L\ddot{o}b} := \Box(\Box A \to A) \to \Box A$.
  Implies $\Box A \to \Box\Box A$. (Due to Dick de Jongh)
- modus ponens: $A, A \to B / B$.
- Necessitation: $A / \Box A$.

Kripke semantics GL

GL is sound and complete for

finite transitive irreflexive Kripke models.

PL(PA^n, PA)

- PA^n is PA plus all true Π^0_n-sentences.
- PA^n is not recursively enumerable.
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$$\begin{aligned}\mathsf{PL}(\mathsf{PA}^n, \mathsf{PA}) :=& \text{Provability logic of } \mathsf{PA}^n \text{ in } \mathsf{PA} \\ :=& \{A \in \mathcal{L}_\Box : \forall \sigma\ \mathsf{PA} \vdash \sigma_{\mathsf{PA}^n} A\}\end{aligned}$$

- $\mathsf{PL}(\mathsf{PA}^n, \mathsf{PA}) := \mathsf{GL}$.

All PA^n's together

- What happens if we consider all PA^n's provability predicates at once?
- We extend first the propositional language.
- We add a unary modal operator $[n]$ for every $n \in \omega$. $(\mathcal{L}_\omega)$
- What are all arithmetically valid $A \in \mathcal{L}_\omega$?

GLP: Japaridze 1986

The system GLP axiomatized over $\mathcal{L}_\omega$:

- All axioms of classical logic.
- All axioms of GL for every $[n]$.
- Monotonicity: $[n]A \to [m]A$ for $m > n$.
- Π^0_{n+1}-completeness: $\neg[n]A \to [m]\neg[n]A$ for $m > n$.
- Modus ponens: $\dfrac{A \qquad A \to B}{B}$.
- Necessitation: $\dfrac{A}{[n]A}$.

Theorem (Japaradize)

GLP $\vdash A$ *iff* PA $\vdash \alpha_\omega(A)$ *for every* α.

GLP is . . .

- interesting due to its applications in the study of proof-theoretic strength of theories.
- not sound and complete for any finite Kripke frames.
- sound and complete for Topological semantics.

Tow valid theorems

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J: a subsystem of GLP

J := GLP with monotonicity replaced by:

- Σ^0_{n+1}-completeness: $[n]A \to [m][n]A$ for $m > n$.
- Weak monotonicity: $[m]A \to [m][n]A$ for $m > n$.

About J

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Theorem (Beklemishev 2011)

GLP $\vdash A$ *iff* J $\vdash M^+(A) \to A$.

$$M(A) := \bigwedge_{[n]B \in A,\ n<i<N} ([n]B \to [n+i]B)$$

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Idea of proof.

J $\vdash M^+(A) \to A \Longrightarrow$ GLP $\vdash A$ is easy.
The other way is proved by arithmetical completeness:

$$\mathsf{J} \nvdash M^+(A) \to A \Longrightarrow \mathsf{PA} \nvdash \alpha_\omega(M^+(A) \to A).$$

❑

GLJ: a subsystem of GLP and J

GLJ := GLP with monotonicity replaced by:

- Σ^0_{n+1}-completeness: $[n]A \to [m][n]A$ for $m > n$.

GLJ: motivating observation

The following two axioms hold in GLJ:

- Σ^0_{n+1}-completeness: $[n]A \to [m][n]A$.
- Π^0_{n+1}-completeness: $\neg[n]A \to [m]\neg[n]A$ for $m > n$.

Together they reduce decision for $[n]$ within $[m]$ to outside of $[m]$.

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- GLJ is sound and complete for finite *Kripke-style models*.

Kripke semantic for GLJ

- GLJ is sound and complete for finite *Kripke-style models*.
- $\mathcal{K} = (W, \{\prec_n\}_{n\in\omega}, V)$.
- Every node (world) has a unique predecessor.
- Some $\prec_n$-accessible world w, validates $[n+i]A$ iff every $w \prec_{n+i} u$ validates A.
- Some $\prec_n$-accessible world w, validates $[n-i-1]A$ iff the predecessor of w validates A.

Translation: first step

Theorem (M. 2024?!)

$\mathsf{GLP} \vdash A$ *iff* $\mathsf{GLJ} \vdash A^+ \to A$.

A^+ is the conjunctions of all formulas of the following shape for every $[i]C \in A$ and $0 \leq j_1 < j_2 < \ldots < j_l \leq i < k \leq r(A)$:

$$[j_1][j_2]\ldots[j_l]([i]C \to [k]C)$$

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The other way is proved by arithmetical completeness:

$$\mathsf{GLJ} \nvdash A^+ \to A \Longrightarrow \mathsf{PA} \nvdash \alpha_\omega(A^+ \to A).$$

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About GLJ

GLJ is well-behaving and looks natural.

Theorem (M.)

GLJ *is sound and complete for poly-interpretations.*

- A poly-interpretation is a substitution α together with a sequence $\{T_n\}_{n\in\omega}$ of theories which are Π^0_n-complete and Σ^0_{n+1}-axiomatizable.

Translation: the second step

- We define a translation $(.)^{\mathsf{j}}$ from GLJ to GL_ω.
- The idea of translation $(.)^{\mathsf{j}}$ is simple: It just pulls out every non-increasing nested boxes.
- Example: $([2]([1]p \to q))^{\mathsf{j}} := [1]p \to [2]q$.
- Nevertheless, a precise general definition needs to take care of many details.

$$X^n([m]B) := \begin{cases} \top & : [m]B \in X \text{ and } m < n, \\ \bot & : [m]B \notin X \text{ and } m < n, \\ [m]X^n(B) & : m \geq n. \end{cases}$$

$$X_n := \{[m]B \in X : m < n\}.$$

$$Y := \{B \in \mathcal{L}_\omega : B \leq A\}$$

$$([n]E)^{\mathfrak{j}} := \bigvee_{X \subseteq_e Y} \left([n]X^n(E^{\mathfrak{j}}) \wedge \bigwedge X_n \wedge \neg \bigvee (Y \setminus X)_n \right).$$

Define $B \leq A$ as the minimum relation satisfying:

- $\leq$ is transitive.
- $B \in \mathsf{sub}(A)$ implies $B \leq A$.
- $\ldots \leq A$ is closed under Boolean operators: $\bot \leq A$ and if $B, C \leq A$ then $\neg B, B \vee C, B \to C, B \wedge C \leq A$.
- $B \leq A$ implies $[n]B \leq [n]A$.

Second step translation

Theorem (M.)

$\mathsf{GLJ} \vdash A$ *iff* $\mathsf{GL}_\omega \vdash A^{\mathsf{j}}$.

Idea of proof.

We show by Kripke semantical argument that $\mathsf{GLJ} \vdash A$ iff $\mathsf{GL}_\omega \vdash A$ for every monotone A. ❑

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Let $A^* := (A^+ \to A)^{\mathsf{j}}$. Then we have:

Corollary

$\mathsf{GLP} \vdash A$ *iff* $\mathsf{GL}_\omega \vdash A^*$.

What is next?

- The variable-free fragment of a subsystem of GLP, called reflection calculus is important due to its applications in proof theory. What is the corresponding subsystem of GLJ?
- The intuitionistic version of poly-modal provability. We know (in a joint work with F. Pakhomov) that HA^n, extension of HA by all true Π^0_n-sentences, has the same provability logic as HA.
- Recently (2024), Beklemishev showed that GLP has nullary unification type. What is the unification type of GLJ? Projective formulas? Admissible rules?

Thanks For Your Attention