On the provability logic of Heyting Arithmetic

Mojtaba Mojtahedi (Ghent University)

June 9, 2023

Provability Logic

- $\Box$ interpreted as provability.
- Gödel 1933: Based on BHK.

Provability Logic: more precise

$$\mathsf{PL}(T) := \text{Provability logic of } T := \{A \in \mathcal{L}_\Box : \forall \sigma\ T \vdash \sigma_T A\}$$

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- $\sigma_T(p) := \sigma(p)$ for atomics.
- σ_T commutes with boolean connectives.
- $\sigma_T(\Box A) := \mathsf{Pr}_T(\ulcorner \sigma_T A \urcorner)$.

What is the Provability logic of PA?

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The Provability logic of PA is GL

- All theorems of classical propositional logic.
- $\mathsf{K} := \Box(A \to B) \to (\Box A \to \Box B)$.
- $\mathsf{L\ddot{o}b} := \Box(\Box A \to A) \to \Box A$.
  Implies $\Box A \to \Box\Box A$. (Due to Dick de Jongh)
- modus ponens: $A, A \to B / B$.
- Necessitation: $A / \Box A$.

Kripke semantics GL

GL is sound and complete for

finite transitive irreflexive Kripke models.

Provability logic of HA

- A. Visser 1980 first considered this problem.
- Since then many partial related results where obtained.
- Main source for difficulty: admissible rules (HA-verifiable).

$$\frac{\neg A \to (B \vee C)}{(\neg A \to B) \vee (\neg A \to C)}$$

Admissible rules

- $A \mathrel{|\!\sim_{\mathsf{T}}} B$ iff $\forall \alpha\ (\mathsf{T} \vdash \alpha(A) \Rightarrow \mathsf{T} \vdash \alpha(B))$.
- Example: $\neg A \to (B \vee C) \mathrel{|\!\sim_{\mathsf{IPC}}} (\neg A \to B) \vee (\neg A \to C)$.
- In the provability logic of HA, the above rule reflected as:

$$\Box(\neg A \to (B \vee C)) \to \Box((\neg A \to B) \vee (\neg A \to C)).$$

- Why not classically interesting?

$$A \mathrel{|\!\sim_{\mathsf{CPC}}} B \quad \text{iff} \quad \mathsf{CPC} \vdash A \to B.$$

Admissible rules of IPC

- For every $A \mathrel{|\!\!\sim_{\mathsf{IPC}}} B$ we have $\Box A \to \Box B$ in $\mathsf{PL}(\mathsf{HA})$.
- What are the admissible rules of IPC? Decidable? (H. Friedman 1975)

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- For every $A \mathrel{|\!\!\sim_{\mathsf{IPC}}} B$ we have $\Box A \to \Box B$ in $\mathsf{PL}(\mathsf{HA})$.
- What are the admissible rules of IPC? Decidable?
  (H. Friedman 1975)
  Decidability: Rybakov 1987.
  Axiomatization: Visser and de Jongh (unpublished).
  Completeness proof: Iemhoff 2001.

The system $[\![\mathsf{T}, \Delta]\!]$

Axioms: Define $A \xrightarrow{\Delta} E := \begin{cases} E & : E \in \Delta \\ A \to E & : \text{otherwise} \end{cases}$

$$\frac{\mathsf{T} \vdash A \to B}{A \rhd B} \; [\mathsf{T}]$$

$$\frac{A = \bigwedge_{i=1}^{n}(E_i \to F_i) \qquad B = \bigvee_{i=n+1}^{n+m}(E_i)}{(A \to B) \rhd \bigvee_{i=1}^{n+m}(A \xrightarrow{\Delta} E_i)} \; \mathsf{V}(\Delta)$$

Rules:

$$\frac{A \rhd B \qquad A \rhd C}{A \rhd (B \wedge C)} \; \text{Conj} \qquad \frac{A \rhd B \qquad B \rhd C}{A \rhd C} \; \text{Cut}$$

$$\frac{A \rhd C \qquad B \rhd C}{(A \vee B) \rhd C} \; \text{Disj} \qquad \frac{A \rhd B \qquad (D \in \Delta)}{(D \to A) \rhd (D \to B)} \; \text{Mont}(\Delta)$$

Admissible Rules of IPC

Theorem (Iemhoff 2001)

$A \mathrel{|\!\!\sim}_{\mathsf{IPC}} B$ *iff* $[\![\mathsf{IPC}, \mathsf{cons}]\!] \vdash A \rhd B$.

Theorem (Visser 2002)

$A \mathrel{|\!\!\sim}_{\mathsf{IPC}} B$ *iff* $[\![\mathsf{IPC}, \mathsf{cons}]\!] \vdash A \rhd B$ *iff* $\Box A \to \Box B \in \mathsf{PL}(\mathsf{HA})$.

What else in $\mathsf{PL}(\mathsf{HA})$? (Disjunction property)

- DP means that if a disjunction is derivable, then either of them are derivable.
- IPC, IQC and HA has DP.
- $\mathsf{CPC} \vdash p \vee \neg p$ while $\mathsf{CPC} \nvdash p$ and $\mathsf{CPC} \nvdash \neg p$.
- $\Box(A \vee B) \to (\Box A \vee \Box B) \in \mathsf{PL}(\mathsf{HA})$?
- H. Friedman 1975: No!
- D. Leivant 1975: $\Box(A \vee B) \to \Box(\boxdot A \vee \boxdot B) \in \mathsf{PL}(\mathsf{HA})$.

What else in $\mathsf{PL}(\mathsf{HA})$? (Markov Rule)

$$\forall S \in \Sigma_1 \ \big(\mathsf{HA} \vdash \neg\neg S \quad \text{implies} \quad \mathsf{HA} \vdash S\big).$$

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Theorem (Visser 1981)

$$\Box\neg\neg\Box A \to \Box\Box A \in \mathsf{PL}(\mathsf{HA}).$$

Theorem (Visser 1981)

The letterless fragment of $\mathsf{PL}(\mathsf{HA})$ *is decidable.*

PL(HA): the axiomatization

Let us define the Leivant's axiom schema as follows:

$$(\mathsf{Le}):\ A \rhd \Box A \text{ for every } A \text{ and } B.$$

Theorem (M. 2022)

$\mathsf{iGLH} := \mathsf{iGL} + \{\Box A \to \Box B : [\![\mathsf{iGL}, \Box]\!]\mathsf{Le} \vdash A \rhd B\} = \mathsf{PL}(\mathsf{HA})$.

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Theorem (Ardeshir & M. 2018)

$\mathsf{iGLC_aH_\sigma} := \mathsf{iGLC_a} + \{\Box A \to \Box B : [\![\mathsf{iGLC_a}, \mathsf{atomb}]\!]\mathsf{Le} \vdash A \rhd B\} = \mathsf{PL}_\Sigma(\mathsf{HA})$

Arithmetical soundness

The arithmetical soundness of this system in a more general setting, namely Σ_1-preservativity, was already known to Visser, de Jongh and Iemhoff (2001).

Completeness: Previous major contributions

1. The NNIL-propositions and NNIL-algorithm. (Visser)
2. Visser rules and its soundness for arithmetical interpretations. (Visser)
3. Unification type of intuitionistic logic. (Ghilardi)
4. Admissible Rules of Intuitionistic Logic. (Iemhoff)
5. The Σ_1-provability logic of HA. (Ardeshir & M.)

Arithmetical Completeness of iGLH

Two *new tools* introduced:

1. Mixed semantics.
2. Relativised unification and admissibility.

Mixed Semantics

For the classical provability:

$$w \Vdash \Box A \quad \text{iff} \quad \forall u \geq w (u \Vdash A).$$

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Since intuitionistic provability requires more witnesses than classical, here in the semantics we need to strengthen:

$$w \Vdash \Box A \quad \text{iff} \quad \forall u \geq w \ (\Gamma_u \vdash A).$$

$\Gamma_u :=$ a fragment of valid propositions at the node u.

Γ-fication: a generalization for unification

- Unification problem:

$$\text{Given } A \text{ find all subs } \theta \text{ such that } \vdash \theta(A).$$

- Γ-fication:

$$\text{Given } A \text{ find all subs } \theta \text{ such that } \vdash \theta(A) \leftrightarrow B \text{ for some } B \in \Gamma.$$

Projectivity

Silvio Ghilardi showed that projectivity is essential for unification problem.

Projectivity: standard (Ghilardi) definition

A is projective iff there is some θ s.t. $\mathsf{IPC} \vdash \theta(A)$ and $A \vdash_{\mathsf{IPC}} \theta(x) \leftrightarrow x$ for every variable x.

Theorem

A projective unifier is a most general unifier.

Proof.

Consider some α s.t. $\mathsf{IPC} \vdash \alpha(A)$. Then $\alpha(A) \vdash \alpha\theta(x) \leftrightarrow \alpha(x)$. This means that $\alpha\theta = \theta$, hence θ is more general than α. ❑

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Theorem (Ghilardi 1999)

For every A there is a finite set of projective propositions $\Pi(A)$ such that $\mathsf{IPC} \vdash \bigvee \Pi(A) \to A$ and $A \mathrel{|\!\!\sim}_{\mathsf{IPC}} \bigvee \Pi(A)$.

NNIL(par)-projectivity

A is NNIL(par)-projective if there is some θ and $B \in \mathsf{NNIL}(\mathsf{par})$ s.t. $\mathsf{IPC} \vdash \theta(A) \leftrightarrow B$ and $A \vdash_{\mathsf{IPC}} \theta(x) \leftrightarrow x$ for every var x.

Theorem (M. 2022)

For every A there is a finite set of NNIL(par)-projective propositions $\bigvee \Pi(A)$ such that $\mathsf{IPC} \vdash \bigvee \Pi(A) \to A$ and

$$A \mathrel{|\!\!\sim}^{\mathsf{N(par)}}_{\mathsf{IPC}} \bigvee \Pi(A)$$

Theorem (M. 2022)

$A \mathrel{|\!\!\sim}^{\mathsf{N(par)}}_{\mathsf{IPC}} B$ *iff* $[\![\mathsf{IPC}, \mathsf{par}]\!] \vdash A \rhd B$.

Future works

- Provability logic of HA relative in other systems.
- Provability logic of extensions of HA by Markov Principle or/and Extended Church Thesis. Specifically: Propositional Logic of $\mathsf{HA} + \mathsf{MP} + \mathsf{ECT}$, and closed fragment of provability logic of $\mathsf{HA} + \mathsf{ECT}$.
- Provability logic of subsystems of HA. (Visser & Miranda)
- Classification of all intuitionistic provability logics? (The intuitionistic version of BAJV theorem)
- Preservativity logic of Heyting Arithmetic. (Iemhoff's Conjecture)
- Study of relative unification and admissibility for other logics and sets other than NNIL.
- Mixed semantics for other intuitionistic modal systems.
- Development of Visser's variant of mixed semantics.

Mojtahedy@gmail.com

http://mmojtahedi.ir/

Thanks For Your Attention