Projectivity meets Uniform Post-Interpolant: Classical and Intutionistic Logic

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Unification

- Given A, unification asks for all substitutions θ s.t.

$$\vdash \theta(A)$$

- If θ unifies A then $\lambda\theta$ also unifies it.
- We say θ is more general than γ if there is some λ s.t. $\gamma = \lambda\theta$.
- Complete set of unifiers: a set of unifiers that every unifier is less general than an element of it.

Main Application: Admissible Rules

$$A \mid\!\sim B \quad \text{iff} \quad \forall\theta\,(\vdash \theta(A) \;\Rightarrow \vdash \theta(B)).$$

Example.

$x \wedge (x \to y) \mid\!\sim y$. This usually simplified as

$$\frac{A \qquad A \to B}{B}$$

In this notation the arbitrary substitution θ which $\theta(x) = A$ and $\theta(y) = B$ is implicit.

Harrop 1960

$\neg x \to (y \vee z) \mid\!\sim (\neg x \to y) \vee (\neg x \to z)$.

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Observation

If A has a finite complete set of unifiers, then admissibility is decidable.

Projectivity: A crucial tool

Given A, we say that θ is A-projection if for every variable x

$$A \vdash \theta(x) \leftrightarrow x.$$

Observation.

A-projections are more general than all unifiers of A.

Proof. Let γ unifies A. Then $\gamma(A) \vdash \gamma\theta(x) \leftrightarrow \gamma(x)$ and thus $\vdash \gamma\theta(x) \leftrightarrow \gamma(x)$ for every variable x.

Definition.

A is called projective iff there is an A-projection which unifies it.

Observation:

Every unifiable formula has a one-element complete set of unifiers.

Proof. Let $\vdash \theta(A)$.

- $\epsilon_\theta(x) := (A \wedge x) \vee (\neg A \wedge \theta(x))$.
- ϵ_θ is A-projection.
- $A \vdash \epsilon_\theta(A) \leftrightarrow A$ and then $A \vdash \epsilon_\theta(A)$.
- $\neg A \vdash \epsilon_\theta(x) \leftrightarrow \theta(x)$ then $\neg A \vdash \epsilon_\theta(A) \leftrightarrow \theta(A)$.
- $\neg A \vdash \epsilon_\theta(A)$.
- $\vdash \epsilon_\theta(A)$.

Intuitionistic Logic

$x \vee \neg x$ does not have a most general unifier. All unifiers of it are $\theta(x) := \top$ and $\theta(x) := \bot$.

Theorem (S. Ghilardi 1999)

The unification type of Intuitionistic Logic is finitary, i.e. for every formula there is a finite complete set of unifiers.

Application (R. Iemhoff 2001)

Completeness of a base for admissible rules of Intuitionistic Logic.

Extending language by parameters

- We assume that the language also has a set of atomic constants (parameters).
- x, y for variables and p, q for parameters.
- Substitutions leave parametrs unchanged.
- In CL: Every unifiable formula is projective.
- In IL: Every unifiable formula has a finite complete set of unifiers.

par-projectivity

- $A := p \wedge x$ can not be projective, since it is not unifiable.
- Instead of unifiers, we look for E-fiers for some parametric (variable-free) formula E.
- An E-fier of A is a substitution θ s.t. $\vdash \theta(A) \leftrightarrow E$.
- We say that A is par-projective, if there is some parametric E and A-projection E-fier for A:

$$\vdash \theta(A) \leftrightarrow E \quad \text{and} \quad A \vdash \theta(x) \leftrightarrow x.$$

In this case E is called a par-projection of A.

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Every par-projective formula has a unique par-projection.

Proof. For $i \in \{1, 2\}$ let θ_i be an A-projection E_i-fier of A.

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- $\vdash \theta_2(A) \to E_1$.
- $\vdash E_2 \to E_1$.
- Similarly $\vdash E_1 \to E_2$.

Given A, the Uniform Post-Interpolant of A with respect to par is defined as a formula A^{par} s.t.:

1. A^{par} is parametric,
2. $\vdash A \to A^{\mathsf{par}}$,
3. For every parametric E with $\vdash A \to E$, we have $\vdash A^{\mathsf{par}} \to E$.

It is well-known that CL and IL both have UI.

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- $\vdash A \to E$ and E is parametric.

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Proof. Let θ be an A-projection E-fier of A.

- $A \vdash \theta(A) \leftrightarrow A$. (by A-projectiveness)
- $\vdash A \to E$ and E is parametric.
- Take parametric F s.t. $\vdash A \to F$.
- $\vdash \theta(A) \to F$.
- Thus $\vdash E \to F$.

Theorem (Papafilippou & M.)

Every formula is par*-projective.*

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- A is extendable if every $\mathcal{K} \Vdash^{-} A$ has a variant $\mathcal{K}' \Vdash A$.

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- A is extendable if every $\mathcal{K} \Vdash^{-} A$ has a variant $\mathcal{K}' \Vdash A$.
- I.e. : A is extendable if every finite set of Kripke models of A can be extended from below s.t. it also be a model of A.

Theorem (S. Ghilardi 1999)

A formula is projective iff it is extendable.

par-projectivity for IL

We say that $\mathcal{K}'$ is a par-variant of $\mathcal{K}$ if they share

1. same frame,
2. same valuation for par,
3. same valuation for variables at any world except the root.

We say that A is E-extendable if

- $\vdash A \to E$,
- Every $\mathcal{K} \Vdash^- A$ with $\mathcal{K} \Vdash E$ has a par-variant $\mathcal{K}' \Vdash A$.

Theorem (Papafilippou & M.)

A formula is E-projective iff it is E-extendable.

Connection to standard projectivity

Question

Can we express par-projectivity through standard projectivity?

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Theorem (Papafilippou & M.)

par-projectivity is equivalent to projectivity of $A^{\mathsf{par}} \to A$.

Proof.

Right-to-Left: Take some $(A^{\mathsf{par}} \to A)$-projection θ that unifies $A^{\mathsf{par}} \to A$. The same θ is also A-projective and A^{par}-fier.

Right-to-Left: Not straightforward. We could prove it separately for CL and IL.

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3. Decidability of Admissible Rules of extensions of intuitionistic logic by parametric axioms.

Theorem (Papafilippou & M.)

The unification type of parametric extensions of IL are finitary.

Relative Admissibility

In the same manner that admissibility relies on standard unification problem, we have relative admissibility, best fit for parametric unification.

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Definition.

$$\mathrel{|\!\sim}_\Gamma := \bigcap_{E \in \Gamma \cap \mathcal{L}(\mathsf{par})} \mathrel{|\!\sim}_E \quad \text{or equivalently:}$$

$$A \mathrel{|\!\sim}_\Gamma B \quad \text{iff} \quad \forall E \in \Gamma \cap \mathcal{L}(\mathsf{par})\ A \mathrel{|\!\sim}_E B$$

Relative Admissibility (continued)

Theorem (Papafilippou & M.)

For every Γ *closed under parameter-substitutions,* $\mid\!\sim_\Gamma$ *is equal to* $\vdash$.

Admissibility relative to NNIL

Theorem (M. 2022)

$\mathrel{|}\!\sim_{\mathsf{NNIL}}$ *is decidable.*

Mojtahedi, Mojtaba. "Relative Unification in Intuitionistic Logic: Towards provability logic of HA." (arXiv 2022).

Mojtahedi, Mojtaba. "On Provability Logic of HA." (arXiv 2022).

Future Works

1. Relative unification and admissibility for transitive modal logic.
2. Axiomatization or decidability of $\mid\!\sim_\Gamma$ for Γ being the set of all extendible formulas.
3. Axiomatization or decidability of $\mid\!\sim_\Gamma$ for Γ being the set of all prime formulas.

Thanks For Your Attention

Intuitionistic Provability Logic

- Problem: Complete axiomatization and decidability of Provability Logic of HA.
- This question was taken up by A. Visser and D. de Jongh and their students since late 70.
- A. Visser 1981: decidability of leterless fragment.
- M. Ardeshir & M. 2018: The Σ-provability logic of HA.
- M. 2022: characterization and decidability of intuitionistic provability logic.