Localizing finite-depth Kripke models

Mojtaba Mojtahedi

University of Tehran

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Intuitionistic Logic (First-order or Propositional)

The intuitionistic Logic is simply Classical logic without the Principle of Excluded Middle (PEM):

$$PEM := A \vee \neg A$$

Or equivalently:

$$\neg\neg A \to A$$

- PA and HA

Kripke Semantics

A Kripke model is a partially ordered set of Classical Structures (Models) $\{\mathfrak{M}_i\}_{i\in I}$ (ordered with $\leq$), with the following property ($i \leq j$):

1. $|\mathfrak{M}_i| \subseteq |\mathfrak{M}_j|$.
2. For each atomic formula A, $\mathfrak{M}_i \models A$ implies $\mathfrak{M}_j \models A$

Kripke Semantics

Define $\mathcal{K}, i \Vdash B$, inductively:

1. $\mathcal{K}, i \Vdash B$ iff $\mathfrak{M}_i \models B$, for atomic B
2. $\mathcal{K}, i \Vdash B_1 \vee B_2$ iff $\mathcal{K}, i \Vdash B_1$ or $\mathcal{K}, i \Vdash B_2$
3. $\mathcal{K}, i \Vdash B_1 \wedge B_2$ iff $\mathcal{K}, i \Vdash B_1$ and $\mathcal{K}, i \Vdash B_2$
4. $\mathcal{K}, i \Vdash B_1 \to B_2$ iff for all $j \geq i$, $\mathcal{K}, j \Vdash B_1$ implies $\mathcal{K}, j \Vdash B_2$
5. $\mathcal{K}, i \Vdash \forall x A(x)$, iff for all $j \geq i$ and $a \in |\mathfrak{M}_j|$, we have $\mathcal{K}, j \Vdash A(a)$
6. $\mathcal{K}, i \Vdash \exists x A(x)$ iff there exists some $a \in |\mathfrak{M}_i|$ such that $\mathcal{K}, i \Vdash A(a)$

Interesting Problem

What is the relationship between $\mathfrak{M}_i \models$ and $\mathcal{K}, i \Vdash$? Specially: does $\mathcal{K}, i \Vdash \mathsf{HA}$ implies $\mathfrak{M}_i \models \mathsf{PA}$?

Finite depth models of HA

D. van Dalen, H. Mulder, E. C. W. Krabbe, and A. Visser, *Finite Kripke models of* HA *are locally* PA, Notre Dame Journal of Formal Logic **27** (1986), no. 4, 528–532.

Other's works

- S. Buss, *Intuitionistic Validity in T-Normal Kripke Structures*, Annals of Pure and Applied Logic **59** (1993) 159-173
- Kai F. Wehmeier, *Classical and intuitionistic models of arithmetic*, Notre Dame J. Formal Logic **37** (1996), no. 3, 452–461.
- M. Ardeshir and B. Hesaam, *Every Rooted Narrow Tree Kripke Model of* HA *is Locally* PA, Mathematical Logic Quarterly **48** (2002), no. 3, 391–395.

Friedman's Translation

- $A^\rho := A \vee \rho$, for atomic formula A, including $\bot$,
- $(A_1 \circ A_2)^\rho := A_1^\rho \circ A_2^\rho$ and $\circ \in \{\vee, \wedge, \rightarrow\}$,
- $(\forall x A)^\rho := \forall x(A^\rho)$,
- $(\exists x A)^\rho := \exists x(A^\rho)$.

Pruning of Kripke models

Let $\mathcal{K} = (K, \leq, \{\mathfrak{M}_i\}_{i \in K})$ be a Kripke model and ρ be a fixed sentence. We can define a new Kripke model, the *pruned* model with respect to ρ, $\mathcal{K}^\rho = (K^\rho, \leq^\rho, \{\mathfrak{M}_i\}_{i \in K^\rho})$, where $K^\rho = K \setminus \{\alpha \in K \mid \alpha \Vdash \rho\}$ and $\leq^\rho$ is the restriction of $\leq$ to the set K^ρ.

Pruning Lemma

Pruning Lemma

Let $\rho \in \mathcal{L}$ be a sentence and $\mathcal{K} = (K, \leq, \{\mathfrak{M}_i\}_{i \in K})$ be a Kripke model for the language $\mathcal{L}$ and $\alpha \in K$ such that $\mathcal{K}, \alpha \nVdash \rho$. Then for all sentences A:

$$\mathcal{K}^\rho, \alpha \Vdash A \text{ iff } \mathcal{K}, \alpha \Vdash A^\rho.$$

Theorem

Suppose $\mathcal{K} = (K, \leq, \{\mathfrak{M}_i\}_{i \in K})$ is a finite-depth Kripke model for the language $\mathcal{L}$. Then for any $\alpha \in K$, there exists some ρ such that for any sentences $A \in \mathcal{L}(\mathfrak{M}_\alpha)$,

$$\mathcal{K}, \alpha \Vdash A^\rho \text{ iff } \mathfrak{M}_\alpha \models A.$$

We say that ρ is a localizer for α in $\mathcal{K}$.

Refinement

Localizers can be restricted to PEM*, the closure of PEM under PEM-Friedman's translations.

Localizers for infinite-depth nodes of Kripke models might not exist.

$$\mathcal{K}, \alpha_0 \Vdash A \text{ iff } \vdash A, \text{ for any } A$$

If ρ is a localizer for α_0:

$$\mathcal{K}, \alpha_0 \Vdash A^\rho \text{ iff } \mathfrak{M}_{\alpha_0} \models A, \text{ for any } A$$

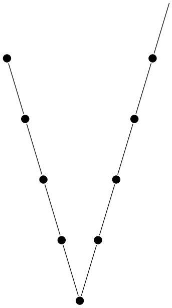
Put $A = \bot$. Then $\alpha_0 \nVdash \rho \quad \Rightarrow \quad \nvdash \rho$.

$$\mathfrak{M}_{\alpha_0} \models A \text{ iff } \vdash A \text{ for atomic } A \text{ (Use DP)}$$

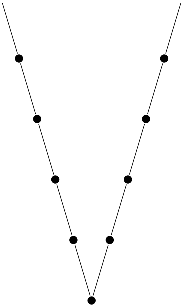
$\mathfrak{M}_{\alpha_0} \models A \to \bot \quad \Rightarrow \quad \vdash A \to \rho \quad \Rightarrow \quad \vdash \rho.$

Narrow and Semi-Narrow

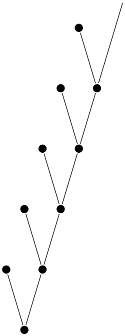
- A Kripke model is narrow if there is no infinite set of pairwise incomparable nodes.
- $d(u)$ indicates the distance of u from leaves,
- We say that a Kripke model is semi-narrow, if for any set of pairwise incomparable nodes X there is some n such that for almost all $u \in X$ (all but finitely many of them), we have $d(u) \leq n$.



"tick" frame



"V" frame



"comb" frame

let $A^{\forall}$ be the formula obtained from A by replacing any $\forall x B$ subformula of A by $\forall x \neg\neg B$

Theorem

For a semi-narrow Kripke model $\mathcal{K} = (K, \leq, \{\mathfrak{M}_i\}_{i \in K})$ with tree frame for a language $\mathcal{L}$ and any $\alpha \in K$, there exists some $\rho \in \mathsf{PEM}_{\mathsf{sen}}(\mathcal{L}(\mathfrak{M}_\alpha))^*$ such that for all sentences $A \in \mathcal{L}(\mathfrak{M}_\alpha))$,

$$\mathcal{K}, \alpha \Vdash (A^{\forall})^{\rho} \quad \text{iff} \quad \mathfrak{M}_\alpha \models A$$

PEM_1

$h(A)$ is the minimum number n, such that there exists some depth-n Kripke model refuting A. Let

$$\mathsf{PEM}_i := \{A \in \mathsf{PEM} : h(A) = i\}$$

We have

$$\mathsf{PEM} = \mathsf{PEM}_\infty \sqcup \mathsf{PEM}_\omega \sqcup \bigsqcup_{k\in\mathbb{N}} \mathsf{PEM}_k$$

Define

- $A_0 := \bot$,
- $A_{n+1} := p_n \vee (p_n \to A_n)$.

For all $n \in \mathbb{N}$ we have $A_i \vee \neg A_i \in \mathsf{PEM}_i$.

Theorem

Localizers for any finite-depth Kripke model could be chosen from PEM_1^*, the closure of PEM_1 under PEM_1-Friedman's translations.

Question

Is there any set $\mathsf{X} \subsetneq \mathsf{PEM}_1$ such that localizers could be chosen from X^*?

Corollary 1

If T is closed under Friedman's translation $(.)^{\rho}$, for any $\rho \in \mathsf{PEM}_1$, Then any finite-depth Kripke model of T is locally T.

Corollary 2

If T is closed under the translation $()^{\forall}$ and PEM-Friedman's translation. Then any semi-narrow Kripke model of T with tree frame is locally T.

Burr's Fragments

Burr's classes Φ_n of formulas in $\mathcal{L}_{\mathsf{a}}$:

- $\Phi_0 := \{A \in \mathcal{L}_{\mathsf{a}} : \ A \text{ is open}\}$,
- $\Phi_1 := \{\exists \bar{x}\, A : \ A \in \Phi_0\}$,
- $\Phi_n := \{\forall \bar{x}(B \to \exists \bar{y} C) : \ B \in \Phi_{n-1}, \ C \in \Phi_{n-2}\} \cup \Phi_{n-1}$, for $n \geq 2$.

W. Burr, *Fragments of Heyting Arithmetic*, Journal of Symbolic Logic **65** (2000), no. 3, 1223–1240.

- Every formula in $\mathcal{L}_{\mathsf{a}}$ is equivalent (in $i\Sigma_1$) to a formula in some Φ_n,
- For $n \geq 2$, every formula in Φ_n is classically equivalent to some Π_n formula,
- For every $n \geq 2$, $\mathrm{I}\Pi_n$ is Π_2-conservative over $i\Phi_n$.

Corollary 3

Burr's hierarchies of HA, $i\Phi_n$ are not closed under PEM_1-Friedman's translation

Thanks