Provability Interpretations for intuitionistic modal logic

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- *Theory*:
  - Consistent and first-order.
  - Finite set of axiom schemes, or decidable axiom set.
- *Strong enough*: It must include basic number theory.

Arithmetization of Syntax.

Main Feature of Gödel's Incompleteness Theorems

Arithmetization of Syntax.

Speaking about provability in T within the same theory T.

Modal Logic: Magical Box

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- Intuitive interpretations: various. Knowledge, Temporal, Obligation, Belief, Program verification and more.
- Main challenge: What is the set of valid propositions in a fixed paradigm?
- Endless objections without conclusion, mainly due to imperciseness in the interpretation.

- $\Box$ as provability.
- Gödel 1933: Based on BHK, he introduces a translation from Intuitionistic Logic to $\mathsf{S4}$.

Why it is interesting?

From a philosophical point of view, provability logic is interesting because:

- The concept of provability in a fixed theory of arithmetic has a unique and non-problematic meaning, other than concepts like necessity and knowledge studied in modal and epistemic logic. Quine was a proponent of syntactical approach to the modal logic.
- Provability logic provides tools to study the notion of self-reference.

Provability Logic: more precise

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- $\sigma_{_T}(p) := \sigma(p)$ for atomics.
- $\sigma_{_T}$ commutes with boolean connectives.
- $\sigma_{_T}(\Box A) := \mathsf{Pr}_{_T}(\ulcorner \sigma_{_T} A \urcorner)$.

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$$\Box(\neg\Box\bot) \to \Box\bot$$

Löb 1955 and Solovay 1976

The Provability logic of *Classical Mathematics* is GL

- All theorems of classical propositional logic.
- $\mathsf{K} := \Box(A \to B) \to (\Box A \to \Box B)$.
- $\mathsf{L\ddot{o}b} := \Box(\Box A \to A) \to \Box A$. Implies $\Box A \to \Box\Box A$.
- modus ponens: $A, A \to B / B$.
- Necessitation: $A / \Box A$.

GL is sound and complete for

finite transitive irreflexive Kripke models.

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- $\mathsf{GL} \nvdash A$.
- $\mathcal{K}, w \not\models A$.
- $f(n) \prec f(n+1)$ iff $n+1$ proves this fact that f will not remain at $f(n+1)$.
- $\sigma(p) := \bigvee_{w \models p}(\lim f = w)$.
- $\mathsf{PA} \nvdash \sigma_{T}(A)$.

$$\mathsf{PL}_{\Sigma}(T) := \Sigma_1\text{-Provability logic of } T := \{A \in \mathcal{L}_\Box : \forall \sigma\ T \vdash \sigma_T A\}$$

Theorem (Visser)

$\mathsf{PL}_{\Sigma}(\mathsf{PA}) = \mathsf{GLC_a} := \mathsf{GL} + p \to \Box p$ *for atomic* p*'s.*

Proof.

Similar to the original proof, except for

- $\sigma(p) := \bigvee_{w \models p} (\exists x f(x) = w)$.

$\square$

Reduction of provability logics

Theorem (Ardeshir & M. 2015)

One may reduce the arithmetical completeness of GL *to the one for* GLCa.

Proof.

Let $\mathsf{GL} \nvdash A$. Then find a Kripke counter model of A. Then transform it to a Kripke model of $\mathsf{GLC_a}$ which refutes $\alpha(A)$ for some propositional substitution α. Thus $\mathsf{GLC_a} \nvdash \alpha(A)$. Finally use arithmetical completeness of $\mathsf{GLC_a}$ and obtain σ such that $\mathsf{PA} \nvdash \sigma\alpha(A)$. ❑

Generalizations

- Relative provability logics: $\mathsf{PL}(T, S)$.
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- Poly-modal Provability Logic.
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- Interpretability logic. $A \rhd B$
  Visser, Berarducci, de Jongh, Veltman, Shavrukov and . . . (1980-1990)
- Provability logic of weak systems of arithmetic (bounded arithmetic).

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- Provability logic of weak systems of arithmetic (bounded arithmetic).
- Provability logic of Heyting Arithmetic HA.

Provability logic of HA

- A. Visser 1980 first considered this.
- Since then many partial related results were obtained. We review them later.
- Main source for difficulty: admissible rules.

$$\frac{\neg A \to (B \vee C)}{(\neg A \to B) \vee (\neg A \to C)}$$

Admissible rules

- $A \mathrel{|\!\sim_{\mathsf{T}}} B$ iff $\forall \alpha\ (\mathsf{T} \vdash \alpha(A) \Rightarrow \mathsf{T} \vdash \alpha(B))$.
- Example: $\neg A \to (B \vee C) \mathrel{|\!\sim_{\mathsf{IPC}}} (\neg A \to B) \vee (\neg A \to C)$.
- In the provability logic of HA, the above rule reflected as:

$$\Box(\neg A \to (B \vee C)) \to \Box((\neg A \to B) \vee (\neg A \to C)).$$

- Why not classically interesting?

$$A \mathrel{|\!\sim_{\mathsf{CPC}}} B \quad \text{iff} \quad \mathsf{CPC} \vdash A \to B.$$

Admissible rules of IPC

- For every $A \mathrel{|\!\!\sim_{\mathsf{IPC}}} B$ we have $\Box A \to \Box B$ in $\mathsf{PL}(\mathsf{HA})$.
- What are the admissible rules of IPC? Decidable? (H. Friedman 1975)

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  (H. Friedman 1975)
  Decidability: Rybakov 1997.
  Axiomatization: Visser and de Jongh (??).
  Completeness proof: Iemhoff 2001.

The system $[\![\mathsf{T}, \Delta]\!]$

Axioms: Define $\{A\}_\Delta(E) := \begin{cases} E & : E \in \Delta \\ A \to E & : \text{otherwise} \end{cases}$

$$\frac{\mathsf{T} \vdash A \to B}{A \rhd B}\ [\mathsf{T}]$$

$$\frac{A = \bigvee_{i=1}^{n}(E_i \to F_i) \qquad B = \bigvee_{i=n+1}^{n+m}(F_i)}{(A \to B) \rhd \bigvee_{i=1}^{n+m}\{A\}_\Delta(E_i)}\ \mathsf{V}(\Delta)$$

Rules:

$$\frac{A \rhd B \qquad A \rhd C}{A \rhd (B \wedge C)}\ \text{Conj} \qquad \frac{A \rhd B \qquad B \rhd C}{A \rhd C}\ \text{Cut}$$

$$\frac{A \rhd C \qquad B \rhd C}{(A \vee B) \rhd C}\ \text{Disj} \qquad \frac{A \rhd B \qquad (D \in \Delta)}{(D \to A) \rhd (D \to B)}\ \text{Mont}(\Delta)$$

Admissible Rules of IPC

Theorem (Iemhoff 2001)

$A \mathrel{|\!\!\sim_{\mathsf{IPC}}} B$ *iff* $[\![\mathsf{IPC}, \mathsf{cons}]\!] \vdash A \rhd B$.

Theorem (Visser 2002)

$A \mathrel{|\!\!\sim_{\mathsf{IPC}}} B$ *iff* $[\![\mathsf{IPC}, \{\top, \bot\}]\!] \vdash A \rhd B$ *iff* $\Box A \to \Box B \in \mathsf{PL}(\mathsf{HA})$.

What else in $\mathsf{PL}(\mathsf{HA})$? (Disjunction property)

- DP means that if a disjunction is derivable, then either of them are derivable.
- IPC, IQC and HA has DP.
- $\mathsf{CPC} \vdash p \vee \neg p$ while $\mathsf{CPC} \nvdash p$ and $\mathsf{CPC} \nvdash \neg p$.
- $\Box(A \vee B) \to (\Box A \vee \Box B) \in \mathsf{PL}(\mathsf{HA})$?
- H. Friedman 1975: No!
- D. Leivant 1975: $\Box(A \vee B) \to \Box(\boxdot A \vee \boxdot B) \in \mathsf{PL}(\mathsf{HA})$.
- Above axiom together with reflection implies DP.

What else in $\mathsf{PL}(\mathsf{HA})$? (Markov Rule)

$$\forall S \in \Sigma_1 \; \big(\mathsf{HA} \vdash \neg\neg S \quad \text{implies} \quad \mathsf{HA} \vdash S\big).$$

What else in $\mathsf{PL}(\mathsf{HA})$? (Markov Rule)

$$\forall S \in \Sigma_1 \ \left(\mathsf{HA} \vdash \neg\neg S \quad \text{implies} \quad \mathsf{HA} \vdash S\right).$$

Theorem (Visser 1981)

$$\Box\neg\neg\Box A \to \Box\Box A \in \mathsf{PL}(\mathsf{HA}).$$

Theorem (Visser 1981)

The letterless fragment of $\mathsf{PL}(\mathsf{HA})$ *is decidable.*

PL(HA): the axiomatization

Let us define the Leivant's axiom schema as follows:

$$(\mathsf{Le}):\ A \rhd \Box A \text{ for every } A \text{ and } B.$$

Theorem (M. 2022)

$\mathsf{iGLH} := \mathsf{iGL} + \{\Box A \to \Box B : [\![\mathsf{iGL}, \Box]\!]\mathsf{Le} \vdash A \rhd B\} = \mathsf{PL}(\mathsf{HA})$.

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Theorem (Ardeshir & M. 2018)

$\mathsf{iGLC_aH}_\sigma := \mathsf{iGLC_a} + \{\Box A \to \Box B : [\![\mathsf{iGLC_a}, \mathsf{atomb}]\!]\mathsf{Le} \vdash A \rhd B\} = \mathsf{PL}_\Sigma(\mathsf{HA})$

Arithmetical soundness

The arithmetical soundness of this system in a more general setting, namely Σ_1-preservativity, was already known by Visser, de Jongh and Iemhoff (2001).

1. Let $\mathsf{iGLH} \nvdash A$.
2. find some α s.t. $\mathsf{iGLC_aH}_\sigma \nvdash \alpha(A)$.
3. use arithmetical completeness of $\mathsf{iGLC_aH}_\sigma$ to find σ s.t. $\mathsf{HA} \nvdash \sigma\alpha(A)$.

Step 2

- We first need a finite, or at least well-behaved Kripke semantics.
- Iemhoff already provided a semantic for an extension of iGLH in the language with binary modal operator.
- Iemhoff's semantics are not finite.
- At least we failed to use it for the purpose of reduction.
- We provided a finite *mixed* semantic which is a combination of derivability and Kripke-style validity.
- It well fits for preservativity.

Two Main Tools

- Mixed Semantics for Modal Logic.

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- Mixed Semantics for Modal Logic.
- Relativising the Notions of Unification and Admissibility.

Thanks For Your Attention