Intuitionistic provability logic: an overview

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- Roughly speaking, provability logic is a modal logic with the intended meaning "provability of A" for $\Box A$.
- The first such interpretation for $\Box$ considered by Gödel in 1933.
- He showed that the intuitionistic propositional logic is interpretable in $\mathsf{S4}$, and hence it is consistent.

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Why provability logic is interesting?

Solovay's completeness 1976

$$\mathsf{PL}(\mathsf{PA}) = \mathsf{GL}$$

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- $\mathsf{PL}(\mathsf{PA},\mathbb{N}) = \mathsf{GLS}$: Solovay 1976.
- $\mathsf{PL}_{\Sigma_1}(\mathsf{PA},\mathsf{PA}) = \mathsf{GL} + \mathsf{C_a}$: Visser 1981.

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- The answer to this question was provided by Beklemishev 1989 on top of previous results due to Artemov, Visser and Japaridze.

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- The question rises here: what is the provability logic for weaker theories like Buss's bounded arithmetic?
- Instead of weakening the induction axiom, one may weaken the background logic.
- What is the provability logic of Heyting Arithmetic (HA)?

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- He showed that some brand new axiom schemata shows up.
  - $\Box(\neg A \to B \vee C) \to \Box[(\neg A \to B) \vee (\neg A \to C)] \in \mathsf{PL}(\mathsf{HA})$.
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- One should expect such axioms because intuitionistic logic has non-derivable admissible rules of inference.

Disjunction property

$$\mathsf{IPC} \vdash A \vee B \quad \Rightarrow \quad \mathsf{IPC} \vdash A \text{ or } \mathsf{IPC} \vdash B$$

Admissible rules of intuitionistic logic

$$A \mathrel{|\!\!\sim_{\mathsf{T}}} B \quad \text{iff} \quad \forall\, \theta(\mathsf{T} \vdash \theta(A) \Rightarrow \mathsf{T} \vdash \theta(B))$$

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$$A \mathrel{|\!\!\sim_{\mathsf{IPC}}} B \quad \text{iff} \quad [\![\mathsf{IPC}, \mathsf{const}]\!] \vdash A \rhd B$$

$\{\mathsf{T}, \Delta\}$: Extending the language to binary modal

T: All theorems and rules of T.

$\mathsf{V}(\Delta)$: $B \to C \rhd \bigvee_{i=1}^{n+m} \{B\}_\Delta(E_i)$, in which $B = \bigwedge_{i=1}^{n}(E_i \to F_i)$ and $C = \bigvee_{i=n+1}^{n+m} E_i$.

Mont(Δ): $A \rhd B \to (C \to A) \rhd (C \to B)$ for every $C \in \Delta$.

Le: $A \rhd \Box A$ for every A.

Disj: $(B \rhd A \wedge C \rhd A) \to (B \vee C) \rhd A$.

Conj: $[(A \rhd B) \wedge (A \rhd C)] \to (A \rhd (B \wedge C))$.

Cut: $[(A \rhd B) \wedge (B \rhd C)] \to (A \rhd C)$.

PNec: $\Box(A \to B) \to (A \rhd B)$.

Σ_1-preservativity

$$A \rhd B : \quad \forall E \in \Sigma_1 \ (\mathsf{HA} \vdash E \to A \Rightarrow \mathsf{HA} \vdash E \to B)$$

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$$\forall\, F \in \Pi_1 \; (\mathsf{HA} \vdash \neg A \to F \Rightarrow \mathsf{HA} \vdash \neg B \to F)$$

$\neg B$ interprets $\neg A$

Propositional preservativity

$$A \mathrel{\underset{\mathsf{T}}{\overset{\Gamma}{\mid\!\approx}}} B \quad \text{iff} \quad \forall\, E \in \Gamma(\mathsf{T} \vdash E \to A \Rightarrow \mathsf{T} \vdash E \to B)$$

Greatest lower bounds

$$\lfloor A \rfloor_{\Gamma}^{\mathsf{T}}$$

Visser [2002]

$$[\![\mathsf{IPC}, \mathsf{atom}]\!] \vdash A \rhd B \quad \text{iff} \quad A \mathrel{|\!\!\approx^{\mathsf{NNIL}}_{\mathsf{IPC}}} B \quad \text{iff} \quad \mathsf{IPC} \vdash \lfloor A \rfloor^{\mathsf{IPC}}_{\mathsf{NNIL}} \to B$$

Visser [2002]

$$[\![\mathsf{IPC}, \mathsf{const}]\!] \vdash A \rhd B \quad \text{iff} \quad A \mathrel{|\!\!\sim_{\mathsf{IPC}}} B \quad \text{iff} \quad A \mathrel{|\!\!\sim_{\mathsf{HA}}} B$$

Projectivity

Parametric version [Mojtahedi, 2022b]

$$[\![\mathsf{IPC}, \mathsf{par}]\!] \vdash A \rhd B \quad \text{iff} \quad A \mathrel{|\!\sim}^{\mathsf{NNIL(par)}}_{\mathsf{IPC}} B \quad \text{iff} \quad A \mathrel{|\!\approx}^{\downarrow\mathsf{NNIL(par)}}_{\mathsf{IPC}} B \quad \text{iff}$$

$$\mathsf{IPC} \vdash {\lfloor A \rfloor_{\mathsf{IPC}}^{\downarrow\mathsf{NNIL(par)}}}^{\vee} \to B$$

$\mathsf{PL}_{\Sigma_1}(\mathsf{HA})$

$$\mathsf{PL}_{\Sigma_1}(\mathsf{HA}) = \mathsf{iGLC_a} + \{\Box A \to \Box B : [\![\mathsf{iGLC_a}, \mathsf{atomb}]\!]\mathsf{Le} \vdash A \rhd B\}$$

Ardeshir and Mojtahedi [2018]

Visser and Zoethout [2019]

PL(HA)

$$\mathsf{PL(HA)} = \mathsf{iGL} + \{\Box A \to \Box B : [\![\mathsf{iGL}, \Box]\!]\mathsf{Le} \vdash A \rhd B\}$$

Mojtahedi [2022a]

Thanks For Your Attention

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