Reduction of Prvability Logics to Σ_1-Provability Logics

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Abstract

We show that the provability logic of PA, GL, and the truth provability logic, i.e., the provability logic of PA relative to the standard model $\mathbb{N}$, GLS are reducible to their Σ_1-provability logics, GLV and GLSV, respectively, by only propositional substitutions.

1 Provability Logic and Σ_1-Provability Logic of PA

Historically, the *Provability Logic of Peano Arithmetic*, PA is discovered [Sol76] before the Σ_1-Provability Logic of PA [Vis81]. The method used in [Vis81] essentially uses Solovay's technique in [Sol76].

The provability logic of PA is a modal logic, well-known as the Gödel-Löb logic, GL that has the following axioms and rules:

- all tautologies of classical propositional logic,
- $\Box(A \to B) \to (\Box A \to \Box B)$,
- $\Box A \to \Box\Box A$,
- Löb's axiom (L): $\Box(\Box A \to A) \to \Box A$,
- Necessitation Rule: $A/\Box A$,
- Modus ponens: $(A, A \to B)/B$.

The *truth provability logic* [AB04] or the *provability logic of* PA *relative to the standard model* $\mathbb{N}$ [BV06], is all of the theorems of GL plus the reflection axiom schema $\Box A \to A$ and the rule of modus ponens. This logic is called GLS [Boo95]. By works of Gödel, for each arithmetical sentence A and sufficiently powerful theory T (like PA), we can formalize the statement "there exists a proof in T for A", by a sentence of the language of arithmetic, i.e. $\exists x \mathsf{Prov}_{\mathsf{T}}(x, \ulcorner A \urcorner)$, where $\ulcorner A \urcorner$ is the code of A. By a T-*interpretation* here, we mean a mapping γ_{T} from the propositional modal language to the first-order language of T, such that

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- $\gamma_{\mathsf{T}}(p)$ is an arithmetical first-order sentence, for any atomic variable p, and $\gamma_{\mathsf{T}}(\bot) = \bot$,
- $\gamma_{\mathsf{T}}(A \circ B) = \gamma_{\mathsf{T}}(A) \circ \gamma_{\mathsf{T}}(B)$, for $\circ \in \{\vee, \wedge, \rightarrow\}$,
- $\gamma_{\mathsf{T}}(\Box A) := \exists x \mathsf{Prov}_{\mathsf{T}}(x, \ulcorner \gamma_{\mathsf{T}}(A) \urcorner)$.

The notion of the Σ_1 T-*interpretation* is defined by restriction of $\gamma_{\mathsf{T}}(p)$ to Σ_1 first-order sentences in the above definition.

Theorem 1.1. [Solovay 1976] *For any sentence A in the language of modal logic,*

1. $\mathsf{GL} \vdash A$ *if and only if for all* PA*-interpretations* γ_{PA}, $\mathsf{PA} \vdash \gamma_{\mathsf{PA}}(A)$,
2. $\mathsf{GLS} \vdash A$ *if and only if for all* PA*-interpretations* γ_{PA}, $\mathbb{N} \models \gamma_{\mathsf{PA}}(A)$.

Let GLV (GLSV) be the Gödel-Löb theory GL (GLS) plus all of the axioms $\mathsf{CP_a} := p \rightarrow \Box p$ in which p is atomic variable, i.e. the completeness principle for atomic propositions.

Theorem 1.2. [Visser 1981] *For any sentence A in the language of modal logic,*

1. $\mathsf{GLV} \vdash A$ *if and only if for all* Σ_1 PA*-interpretations* σ_{PA}, $\mathsf{PA} \vdash \sigma_{\mathsf{PA}}(A)$,
2. $\mathsf{GLSV} \vdash A$ *if and only if for all* Σ_1 PA*-interpretations* σ_{PA}, $\mathbb{N} \models \sigma_{\mathsf{PA}}(A)$.

Proof. See [Vis81] and [Boo95]. ❑

In this paper, we show that the provability logics of PA and the truth provability logic can be obtained from their Σ_1-provability logics by "propositional manipulation". In other words, we reduce the arithmetical completeness of the provability logics of PA and the truth provability logic to the arithmetical completeness of their Σ_1-provability logics. The argument in the reduction process is purely propositional and has nothing to do with the first-order arithmetic. Afterwards we inspect this process in a more general framework, called *the propositional logic of a propositional theory.*

2 Kripke semantics for GL and GLV

The proof of the arithmetical completeness of the provability logic of PA in [Sol76], [Boo95], [Smo85], uses Kripke semantics for GL. For our purpose, in this section, we refine the Kripke completeness theorem.

Let us first clarify some standard notations. A Kripke model is a triple $\mathcal{K} = (K, \prec, V)$ in which $\prec$ is a transitive binary relation over K and V is a relation between elements of K and atomic variables, i.e., $V \subseteq K \times \mathsf{Atom}$. We say that $\mathcal{K}$ is *persistent*, if for all atomic variables p and $k \prec k'$, kVp implies $k'Vp$. We may use the notation $V, k \Vdash p$ instead of kVp. One can extend V to all modal propositions as follows:

- $V, k \nVdash \bot$,
- $V, k \Vdash p$ iff kVp, for atomic variables p,
- $V, k \Vdash A \vee B$ iff "$V, k \Vdash A$ or $V, k \Vdash B$",

- $V, k \Vdash A \wedge B$ iff "$V, k \Vdash A$ and $V, k \Vdash B$",
- $V, k \Vdash A \to B$ iff "$V, k \nVdash A$ or $V, k \Vdash B$",
- $V, k \Vdash \Box A$ iff for all $k' \in K$ such that $k \prec k'$, we have $V, k' \Vdash A$.

We say that $\mathcal{K} = (K, \prec, V)$ is finite (similar notation for trees) if its frame $(K, \prec)$ is finite.

Theorem 2.1. GL *is sound and complete for the class of all finite irreflexive Kripke models.*

Proof. See [Sol76], [Boo95], [Smo85]. ❑

Theorem 2.2. GLV *is sound and complete for all persistent finite irreflexive Kripke models.*

Proof. See [Boo95](page 136). ❑

For any A in the modal language let

$$A^s := \left(\bigwedge_{\Box B \in \mathsf{Sub}(A)} (\Box B \to B) \right) \to A$$

Then we have the following theorem:

Theorem 2.3. *For any A we have* $\mathsf{GL} \vdash A^s$ *iff* $\mathsf{GLS} \vdash A$. *Also* $\mathsf{GLV} \vdash A^s$ *iff* $\mathsf{GLSV} \vdash A$.

Proof. See [Boo95](pages 131 and 136). ❑

Lemma 2.4. *Let α be a propositional substitution such that no $\Box$ is appeared in $\alpha(p)$, for any atomic variables p. Then we have*

$$\alpha(A^s) = (\alpha(A))^s$$

Proof. This could be deduced from the following fact:

$$\star \qquad \Box B \in \mathsf{Sub}(\alpha(A)) \quad \text{iff} \quad B = \alpha(B') \text{ and } \Box B' \in \mathsf{Sub}(A)$$

We prove $(\star)$ by induction on the complexity of A.

- $A = p$ is atomic variable. Then neither $\alpha(A)$ (by use of the hypothesis of Lemma) nor A have any boxed subformula,
- $A = A_1 \circ A_2$ ($\circ \in \{\wedge, \vee, \to\}$) or $A = \neg A_1$: Then $\Box B \in \mathsf{Sub}(\alpha(A))$ iff for some $i \in \{1, 2\}$ we have $\Box B \in \mathsf{Sub}(\alpha(A_i))$ iff (by the induction hypothesis) $B = \alpha(B')$ and $\Box B' \in \mathsf{Sub}(A_i)$ iff $B = \alpha(B')$ and $\Box B' \in \mathsf{Sub}(A)$,
- $A = \Box A_1$. We first reason from left to right. Assume that $\Box B \in \mathsf{Sub}(\alpha(A))$. Then either we have $\Box B = \alpha(A)$ or $\Box B \in \mathsf{Sub}(\alpha(A_1))$. If $\Box B = \alpha(A)$, then $B = \alpha(A_1)$ and $A = \Box A_1 \in \mathsf{Sub}(A)$. If $\Box B \in \mathsf{Sub}(\alpha(A_1))$, then one could reason like the case $A = \neg A_1$. Now we reason from right to left. Assume that $B = \alpha(B')$ and $\Box B' \in \mathsf{Sub}(A)$. Then either $A = \Box B'$ (i.e. $B' = A_1$) or $\Box B' \in \mathsf{Sub}(A_1)$. If $A = \Box B'$, then $\Box B = \alpha(A)$ and hence $\Box B \in \mathsf{Sub}(\alpha(A))$. If $\Box B' \in \mathsf{Sub}(A_1)$ ($B = \alpha(B')$), then one could reason like when $A = \neg A_1$.

❑

3 Reduction of GLV to GL and GLSV to GLS

Our main goal in this section is to reduce the arithmetical completeness of GL (similarly for GLS) to the Σ_1-arithmetical completeness of GLV (GLSV). We first treat the PA-interpretations in PA and then do the same work in the standard model $\mathbb{N}$. By a propositional substitution α, we mean a mapping from the set of atomic variables Atom to the set of formulas. The notation $\alpha(A)$ means the result of substitution of α in proposition A.

Lemma 3.1. *Let* GL $\nvdash A$*, for some modal proposition* A*. Then there is some propositional substitution* α *such that* GLV $\nvdash \alpha(A)$.

Proof. Suppose that GL $\nvdash A$. Then there is some finite irreflexive Kripke model $\mathcal{K} = (K, \prec, V)$ such that $V, k_0 \nVdash A$, for some $k_0 \in K$, [Smo85](Theorem 2.6). For any $k \in K$, let p_k be a fresh variable, i.e., p_k is not appeared in A. We define the Kripke model $\mathcal{K}' = (K, \prec, V')$ as the following.

$$V', k' \Vdash p_k \text{ iff } k \preccurlyeq k' \quad \text{ and } \quad V', k \nVdash p, \text{ for any atomic } p \neq p_k$$

Note that $\mathcal{K}'$ is a finite irreflexive persistent Kripke model.

Now, for every $k \in K$, we define a proposition k^* by

$$k^* := p_k \wedge \bigwedge_{k \prec l} \neg p_l$$

Let α be the propositional substitution defined by

$$\alpha(p) := \bigvee_{k \Vdash p} k^*$$

By induction on the complexity of $B \in \mathsf{Sub}(A)$, we show that for any $k \in K$,

$$V', k \Vdash \alpha(B) \quad \text{iff} \quad V, k \Vdash B$$

All the induction steps are easy and we only show the case where $B = p$ is an atomic variable. First we show that $V', k' \Vdash k^*$ iff $k = k'$. Note that $V', k' \Vdash k^*$ iff $V', k' \Vdash p_k \wedge \bigwedge_{k \prec l} \neg p_l$ iff $V', k' \Vdash p_k$ and $V', k' \nVdash p_l$ for any $l \succ k$ iff $k' \succcurlyeq k$ and $k' \not\succcurlyeq l$ for any $l \succ k$ iff $k' = k$. Now by definition of $\alpha(p)$, we have $V', k' \Vdash \alpha(p)$ iff $V', k' \Vdash k^*$ for some k such that $V, k \Vdash p$ iff $k' = k$ for some k such that $V, k \Vdash p$ iff $V, k' \Vdash p$.

Hence $V', k_0 \nVdash \alpha(A)$. Then by soundness part of Theorem 2.2, we have GLV $\nvdash \alpha(A)$. ❑

Corollary 3.2. *The arithmetical completeness of* GL *is reducible to that of* GLV*, i.e. the completeness part of Theorem 1.1.1 (the "if" part) is reducible to that of Theorem 1.2.1.*

Proof. We prove the "if" part of Theorem 1.1.1 contrapositively. Let GL $\nvdash A$. We should prove that there is some arithmetical PA-interpretation γ_{PA} such that PA $\nvdash \gamma_{\mathsf{PA}}(A)$. From GL $\nvdash A$ and Lemma 3.1, we will have some propositional substitution α such that GLV $\nvdash \alpha(A)$. By completeness part of Theorem 1.2.1, there is some arithmetical PA-interpretation σ_{PA}, such that PA $\nvdash \sigma_{\mathsf{PA}}(\alpha(A))$. Let $\gamma_{\mathsf{PA}} := \sigma_{\mathsf{PA}} \circ \alpha$, i.e. the composition of two substitutions α and σ_{PA}. Hence PA $\nvdash \gamma_{\mathsf{PA}}(A)$. ❑

Lemma 3.3. *Let* $\mathsf{GLS} \nvdash A$*, for some modal proposition* A*. Then there is some propositional substitution* α *such that* $\mathsf{GLSV} \nvdash \alpha(A)$*.*

Proof. Let $\mathsf{GLS} \nvdash A$. Then Theorem 2.3 implies that $\mathsf{GL} \nvdash A^s$. By Lemma 3.1, there exists some propositional substitution α such that $\mathsf{GLV} \nvdash \alpha(A^s)$. By Lemma 2.4, we have $\mathsf{GLV} \nvdash [\alpha(A)]^s$, and again by Theorem 2.3, we have $\mathsf{GLSV} \nvdash \alpha(A)$ as desired. ❑

Corollary 3.4. *The arithmetical completeness of* GLS *is reducible to that of* GLSV*, i.e. the completeness part of Theorem 1.1.2 (the "if" part) is reducible to that of Theorem 1.2.2.*

Proof. We prove the "if" part of Theorem 1.1.2 contrapositively. Let $\mathsf{GLS} \nvdash A$. We should prove that there is some arithmetical PA-interpretation γ_{PA} such that $\mathbb{N} \not\models \gamma_{\mathsf{PA}}(A)$. From $\mathsf{GLS} \nvdash A$ and Lemma 3.3, we will have some propositional substitution α such that $\mathsf{GLSV} \nvdash \alpha(A)$. By completeness part of Theorem 1.2.2, there is some arithmetical PA-interpretation σ_{PA}, such that $\mathbb{N} \not\models \sigma_{\mathsf{PA}}(\alpha(A))$. Let $\gamma_{\mathsf{PA}} := \sigma_{\mathsf{PA}} \circ \alpha$, i.e. the composition of two substitutions α and σ_{PA}. Hence $\mathbb{N} \not\models \gamma_{\mathsf{PA}}(A)$. ❑

4 The Propositional Logic of a propositional Theory

A propositional logic of a first-order theory T is defined as the set of all T-valid formulas, where a propositional formula A is called T-valid iff, for all substitutions α of formulas of the language of T for propositional variables, we have $\mathsf{T} \vdash \alpha(A)$ [dJVV11]. This notion comes from the de Jongh property of the constructive first-order arithmetical theories. D. de Jongh proved that the propositional logic of the Heything Arithmetic, HA is precisely Intuitionistic Propositional Logic, IPC [dJ70]. The propositional logics of some constructive theories stronger than HA and weaker than HA are investigated in e.g., [dJVV11] and [AM14], respectively.

Now we want to generalize this notion to propositional logic of a *propositional (modal or non-modal) theory*. Let T be a propositional (modal or non-modal) theory. The propositional logic of T, $\mathcal{L}(\mathsf{T})$, is defined to be the set of all propositions A (modal or non-modal) such that for all propositional substitution α (modal or non-modal) we have $\mathsf{T} \vdash \alpha(A)$. We may present our results of this paper in this framework. Here are another statements of Corollary 3.2 and Corollary 3.4.

Theorem 4.1. $\mathcal{L}(\mathsf{GLV}) = \mathsf{GL}$ *and* $\mathcal{L}(\mathsf{GLSV}) = \mathsf{GLS}$*.*

Let T be a strong enough theory in the first-order language of arithmetic. Let $\mathcal{PL}(\mathsf{T})$ and $\mathcal{PL}_{\Sigma}(\mathsf{T})$ indicate the provability logic and the Σ_1-provability logic of T, respectively. The above theorem tells that the propositional logics of the two Σ_1-provability logics are their provability logics. A natural question is whether *the above theorem holds for every strong enough theory* T *instead of* PA*?* or more precisely, is it the case that $\mathcal{L}(\mathcal{PL}_{\Sigma}(\mathsf{T})) = \mathcal{PL}(\mathsf{T})$?

We do not know the answer of the above question, however we have the following partial answer.

Theorem 4.2. *Let* T *be a strong enough theory in first-order language of arithmetic. Then*

$$\mathcal{L}(\mathcal{PL}_{\Sigma}(\mathsf{T})) \supseteq \mathcal{PL}(\mathsf{T}), \textit{ or in other words, } \quad \mathcal{L}(\mathcal{PL}_{\Sigma}(\mathsf{T})) \vdash \mathcal{PL}(\mathsf{T})$$

Proof. Let $A \in \mathcal{PL}(\mathsf{T})$. Then for all T-interpretations γ_{T}, we have $\mathsf{T} \vdash \gamma_{\mathsf{T}}(A)$. We should show that $A \in \mathcal{L}(\mathcal{PL}_{\Sigma}(\mathsf{T}))$. That means that for arbitrary propositional substitution α, we should have $\alpha(A) \in \mathcal{PL}_{\Sigma}(\mathsf{T})$. Now, it is enough to show that for arbitrary Σ_1 T-interpretation σ_{T}, we have $\mathsf{T} \vdash \sigma_{\mathsf{T}}(\alpha(A))$. However, the composition of substitutions α and σ_{T}, $\gamma_{\mathsf{T}} := \sigma_{\mathsf{T}}.\alpha$ is a T-interpretation and we already have $\mathsf{T} \vdash \gamma_{\mathsf{T}}(A)$. ❏

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