The de Jongh property for Basic Arithmetic

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Abstract

We prove that Basic Arithmetic, **BA**, has the de Jongh property, i.e., for any propositional formula $A(p_1, \cdots, p_n)$ built up of atoms $p_1, \cdots, p_n$, **BPC** $\vdash A(p_1, \cdots, p_n)$ if and only if for all arithmetical sentences $B_1, \cdots, B_n$, **BA** $\vdash A(B_1, \cdots, B_n)$. The technique used in our proof can easily be applied to some known extensions of **BA**.

1 Introduction

Basic Arithmetic, **BA**, is the basic logic, **BQC**, equivalent of Heyting Arithmetic, **HA**, over intuitionistic logic, **IQC**, and of Peano Arithmetic, **PA**, over classical logic, **CQC**.

Basic Propositional Calculus, **BPC**, has been invented by Visser [16]. For more details and different extensions of **BPC** see, e.g., [2]. **BPC** is extended to first-order logic and *Basic Arithmetic* in [10]. For more details about **BA** see, e.g., [1].

Dick de Jongh proved his original theorem in an unpublished paper, that the propositional logic of Heyting Arithmetic **HA** is **IPC**. de Jongh's argument uses an ingenious combination of Kripke models and realizability, [4]. Since the original theorem of De Jongh, different proofs of the theorem have been invented, and moreover it has been extended to different logics, see, e.g., [5, Sec 2], for a brief overview of the history of de Jongh's theorem for propositional logic.

We prove the de Jongh property for **BPC** and **BA**, i.e., **BPC** is complete with respect to **BA**. In other words, any propositional formula $A(p_1, \cdots, p_n)$ built up of atoms $p_1, \cdots, p_n$, **BPC** $\vdash A(p_1, \cdots, p_n)$ if and only if for all arithmetical sentences $B_1, \cdots, B_n$, we have **BA** $\vdash A(B_1, \cdots, B_n)$. Our proof and the one in [5, Th. 4.1] both use the general method of Smoryński [12] to find a *finite* Kripke model of **HA** from an existing Kripke model by adding a non-standard model of **PA** (with appropriate conditions) as a new root. The main difference between our proof and the other ones is that we make use of Solovay's method as described in [13] to obtain a recursive function, here called *the Solovay function*, on a finite frame of a Kripke model, which provides a tool to simulate propositional Kripke models with a Kripke model of **BA**. Moreover, one of the main ingredients of our proof is a direct

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application of the celebrated **MRDP**-Theorem. We use this Theorem to transform any Σ_1-sentence in the language of arithmetic, which may include universal bounded quantifiers, to a $\exists_1$-sentence that is under control in Kripke model theory of **BA**.

A closer look at our proof of the de Jongh property for **BA** reveals that it would also work for some extensions of **BPC** and their corresponding arithmetical theories. For instance, we mention three extensions of **BPC** and their arithmetical counterparts. The first one is the de Jongh property between **IPC** and **HA**, which has already been proved with different methods [5]. The second one is for *Formal Propositional Logic*, **FPC**, introduced in [16] and its arithmetic, Löb Arithmetic, **LA** [10], and the last one is **EBPC**, and its arithmetic, **EBA**, introduced in [1].

2 Basic Arithmetic

2.1 Axioms, rules and some elementary facts

The language of **BQC** is a little different from the usual one for **IQC**. It was originally axiomatized in sequent notation, i.e., using sequents like $A \Rightarrow B$ where A and B are formulas in the language $\{\vee, \wedge, \rightarrow, \perp, \top, \exists, \forall\}$. Since *modus ponens* is not a rule in **BQC**, a universally quantified formula like $\forall x \forall y\, A$ is different from $\forall xy\, A$. When we write $\forall \mathbf{x}\, (A \rightarrow B)$, we mean $\mathbf{x}$ to be a finite sequence of variables *once* quantified. Beside a set of predicate and function symbols of possibly different finite arity, we also include the binary predicate "=" for equality. Terms, atomic formulas and formulas are defined as usual except for universal quantification: if A and B are formulas, and $\mathbf{x}$ is a finite sequence of variables, then $\forall \mathbf{x}\, (A \rightarrow B)$ is a formula. The concepts of free and bound variables are defined as usual. A *sentence* is a formula with no free variables. An *implication* is a universal quantification $\forall \mathbf{x}\, (A \rightarrow B)$ where $\mathbf{x}$ is the empty sequence. $\neg A$ means $A \rightarrow \perp$. Given a sequence of variables $\mathbf{x}$ without repetitions, $s[\mathbf{x}/\mathbf{t}]$ and $A[\mathbf{x}/\mathbf{t}]$ stand for, respectively, the term and formula that results from substituting the term $\mathbf{t}$ for all free occurrences of the variables of $\mathbf{x}$ in the term s and the formula A. For details, see [10].

Axioms and rules of BQC

In the following list, occurrence of a double horizontal line in a rule means that the rule is reversible.

1. $A \Rightarrow A$,

2. $A \Rightarrow \top$,

3. $\perp \Rightarrow A$,

4. $A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)$,

5. $A \wedge \exists x\, B \Rightarrow \exists x (A \wedge B)$, where x is not free in A,

6. $\top \Rightarrow x = x$,

7. $x = y \wedge A \Rightarrow A[x/y]$, where A is atomic,

8. $\forall \mathbf{x}\,(A \to B) \wedge \forall \mathbf{x}\,(B \to C) \Rightarrow \forall \mathbf{x}\,(A \to C)$,

9. $\forall \mathbf{x}\,(A \to B) \wedge \forall \mathbf{x}\,(A \to C) \Rightarrow \forall \mathbf{x}\,(A \to B \wedge C)$,

10. $\forall \mathbf{x}\,(B \to A) \wedge \forall \mathbf{x}\,(C \to A) \Rightarrow \forall \mathbf{x}\,(B \vee C \to A)$,

11. $\forall \mathbf{x}\,(A \to B) \Rightarrow \forall \mathbf{x}\,(A[\mathbf{x}/\mathbf{t}] \to B[\mathbf{x}/\mathbf{t}])$, where no variable in the sequence of terms $\mathbf{t}$ is bound by a quantifier of A or B,

12. $\forall \mathbf{x}\,(A \to B) \Rightarrow \forall \mathbf{y}\,(A \to B)$, where no variable in $\mathbf{y}$ is free on the left hand side,

13. $\forall \mathbf{y} x\,(B \to A) \Rightarrow \forall \mathbf{y}\,(\exists x\, B \to A)$, where x is not free in A,

14. $\dfrac{A \Rightarrow B, A \Rightarrow C}{A \Rightarrow B \wedge C}$ (double line),

15. $\dfrac{B \Rightarrow A, C \Rightarrow A}{B \vee C \Rightarrow A}$ (double line),

16. $\dfrac{A \Rightarrow B}{A[\mathbf{x}/\mathbf{t}] \Rightarrow B[\mathbf{x}/\mathbf{t}]}$, where no variable in the sequence of terms $\mathbf{t}$ is bound by a quantifier in the denominator,

17. $\dfrac{B \Rightarrow A}{\exists x\, B \Rightarrow A}$ (double line), where x is not free in A,

18. $\dfrac{A \wedge B \Rightarrow C}{A \Rightarrow \forall \mathbf{x}\,(B \to C)}$, where no variable in $\mathbf{x}$ is free in A.

We often write A for $\top \Rightarrow A$. Let Γ be a set of sequents and rules. We say Γ *proves* $A \Rightarrow B$, written $\Gamma \vdash A \Rightarrow B$, when $A \Rightarrow B$ can be obtained, after finitely many applications of the **BQC** rules and the rules of Γ from the **BQC** axioms and the axioms of Γ. Similarly, Γ *proves* the rule

$$R = \frac{A_1 \Rightarrow B_1, \ldots, A_n \Rightarrow B_n}{A \Rightarrow B}$$

written as $\Gamma \vdash R$, when $\Gamma \cup \{A_1 \Rightarrow B_1, \ldots, A_n \Rightarrow B_n\} \vdash A \Rightarrow B$. We say Γ is consistent when $\Gamma \nvdash \bot$. A *theory* is a set of sequents and rules closed under derivability. A theory is called a *sequent theory* if it is axiomatized by a set of sequents. Examples of theories are

- **FQC** $=$ **BQC** $+ (\top \to A) \to A \Rightarrow \top \to A$, the Löb axiom,
- **EBQC** $=$ **BQC** $+ \top \to \bot \Rightarrow \bot$,
- **IQC** $=$ **BQC** $+ \top \to A \Rightarrow A$, and
- **CQC** $=$ **BQC** $+ \{\top \to A \Rightarrow A, A \vee \neg A\}$.

Note that **FQC** + **IQC** and **FQC** + **EBQC** are inconsistent theories.

The non-logical language $\mathcal{L}$ of **BA** is $\{0, S, +, \cdot\}$, where 0 is a constant symbol, S is a unary function symbol for successor, and $+$ and $\cdot$ are binary function symbols for addition and multiplication, respectively.

Axioms and rules of BA

1. $Sx = 0 \Rightarrow \bot$,
2. $Sx = Sy \Rightarrow x = y$,
3. $x + 0 = x$,
4. $x + Sy = S(x + y)$,
5. $x \cdot 0 = 0$,
6. $x \cdot Sy = x \cdot y + x$,
7. $\forall \mathbf{y}x\,(A \to A[x/Sx]) \Rightarrow \forall \mathbf{y}x\,(A[x/0] \to A)$,
8. $\dfrac{A \Rightarrow A[x/Sx]}{A[x/0] \Rightarrow A}$.

Löb's arithmetic, **LA**, was first introduced in [10]. **LA** is defined as **BA** + **FQC**. Similarly, **EBA** is defined as **BA** + **EBQC**. *Basic Propositional Calculus*, **BPC**, is axiomatized by the axioms 1-5, and the quantifier-free forms of the axioms 6-8,

- $(A \to B) \wedge (B \to C) \Rightarrow A \to C$,
- $(A \to B) \wedge (A \to C) \Rightarrow A \to B \wedge C$,
- $(B \to A) \wedge (C \to A) \Rightarrow B \vee C \to A$,

and the rules 14-15, and the quantifier-free form of the rule 18,

- $\dfrac{A \wedge B \Rightarrow C}{A \Rightarrow B \to C}$.

Formal Proposition Logic, **FPC**, is defined as

$$\mathbf{FPC} = \mathbf{BPC} + (\top \to A) \to A \Rightarrow \top \to A,$$

It is worth mentioning that **BPC** and **FPC** are the propositional logics which correspond to modal logics **K4** and **GL** by the Gödel translation τ, respectively [16].

The following logic is another interesting extension of **BPC** which behaves very similar to **IPC** [1].

$$\mathbf{EBPC} = \mathbf{BPC} + \top \to \bot \Rightarrow \bot.$$

2.2 Kripke model theory of BPC

A *Kripke model* for **BPC** is a triple $\mathcal{K} = (K, \prec, \Vdash)$, where as usual, K is a non-empty set of *nodes*, "$\prec$" is a binary transitive relation on K, and "$\Vdash$" is a relation between nodes in K and formulas in **BPC** . The forcing definition $k \Vdash A$ is like in **IPC** [14, Ch. 2.5.2], except for the implication, where $k \Vdash B \to C$ is defined as

$k \Vdash A \to B$ if and only if for all $k' \succ k$, $k' \Vdash A$ implies $k' \Vdash B$.

We extend $\Vdash$ to all sequents and rules. For a sequent $A \Rightarrow B$, it is defined by

$k \Vdash A \Rightarrow B$ if and only if for all $k' \succeq k$, $k' \Vdash A$ implies $k' \Vdash B$.

For a rule $R = \dfrac{A_1 \Rightarrow B_1, \ldots, A_n \Rightarrow B_n}{A \Rightarrow B}$, defined by

$k \Vdash R$ if and only if for all $k' \succeq k$, if $k' \Vdash A_i \Rightarrow B_i$ for all i, $1 \leq i \leq n$, then $k' \Vdash A \Rightarrow B$.

A Kripke model $\mathcal{K} = (K, \prec, \Vdash)$ is called *rooted* if there is a node $k_0 \in K$ such that $k_0 \preceq k$, for all $k \in K$. Also it is called a *tree* Kripke model iff $(K, \prec)$ is a tree.

Note that the relation "$\prec$" in our definition of a Kripke model is not necessarily reflexive, that means that in case $k \prec k'$, we may have either $k \neq k'$ or $k = k'$. Also we use $k \preceq k'$ as shorthand for $k \prec k' \vee k = k'$ and $k \precneqq k'$ as shorthand for $k \prec k' \wedge k \neq k'$.

Theorem 2.1 **BPC**, **EBPC** *and* **FPC** *are sound and complete for the class of finite, finite with reflexive leaves and finite irreflexive Kripke models, respectively.*

Proof. For **BPC** and **FPC**, see [2, Th. 4.6] and [16, Th. 2.2], and for **EBPC**, see [1, Cor. 3.9]. $\dashv$

It is a well-known technique [14, Th. 2.6.8], that every finite rooted Kripke model for **IPC** can be transformed into a finite rooted *tree* Kripke model for **IPC**. It is also proved [2, Cor. 4.2], that **BPC** is complete with respect to irreflexive rooted tree Kripke model. In the following, we show that every finite rooted Kripke model for **BPC** can be transformed into a *finite* rooted tree Kripke model of **BPC**, that implies that **FPC** and **EBPC** are complete for finite rooted Kripke models with tree frames as well.

Theorem 2.2 **BPC**, **FPC** *and* **EBPC** *are complete for Kripke models with finite rooted tree frame, irreflexive finite tree frame and finite tree frame with reflexive leaves, respectively.*

Proof. Our proof is by the standard process of tree unravelling (see [12, Th. 5.3.4]). We show that every finite Kripke model $\mathcal{K} = (K, \prec, \Vdash)$ can be transformed to a finite Kripke model with tree frame that preserves the forcing relation. This fact combined with 2.1, implies the desired completenesses.

We use finite sequences of nodes of $\mathcal{K}$ to construct a new rooted tree Kripke model $\mathcal{K}' = (K', \prec', \Vdash')$. Define

- $K' = \{\langle k_0, \cdots, k_n\rangle \mid k_0 \precneqq k_1 \precneqq \cdots \precneqq k_n \ , \ n \in \mathbb{N}\} \cup \{\langle\rangle\}$,
- $\langle k_0, \cdots, k_n\rangle \prec' \langle l_0, \cdots, l_m\rangle$ iff either $n < m$ and $k_i = l_i$, for $i \leq n$, or $n = m$, $k_i = l_i$, for $i \leq n$, and $k_n \prec l_m$.
- $\langle k_0, \cdots, k_n\rangle \Vdash' p$ iff $k_n \Vdash p$, for atomic p. This means that $\langle\rangle \nVdash' p$ for all atomic p.

We show that for all nonempty sequences $\langle k_0, \cdots, k_n\rangle \in K'$, we have $\langle k_0, \cdots, k_n\rangle \Vdash' B$ iff $k_n \Vdash B$, by induction on the complexity of B. We consider the interesting case where $B = C \to D$.

Let $k_n \Vdash C \to D$. To show $\langle k_0, \cdots, k_n \rangle \Vdash' C \to D$, let $\langle k_0, \cdots, k_n, \cdots, k_{n+m} \rangle \in K'$ and $\langle k_0, \cdots, k_n, \cdots, k_{n+m} \rangle \Vdash C$. Then, by induction hypothesis, $k_{n+m} \Vdash C$, and by definition, $k_n \prec k_{n+m}$. So $k_{n+m} \Vdash D$. By induction hypothesis again, $\langle k_0, \cdots, k_n, \cdots, k_{n+m} \rangle \Vdash D$.

For the other direction, let $\langle k_0, \cdots, k_n \rangle \Vdash' C \to D$. To show $k_n \Vdash C \to D$, let $k_n \prec l$ and $l \Vdash C$. Then either $k_n \neq l$ or $k_n = l$.

If $k_n \neq l$, then $\langle k_0, \cdots, k_n, l \rangle \in K'$. Since $\langle k_0, \cdots, k_n \rangle \prec' \langle k_0, \cdots, k_n, l \rangle$, by induction hypothesis, we have $\langle k_0, \cdots, k_n, l \rangle \Vdash' C$, and so $\langle k_0, \cdots, k_n, l \rangle \Vdash' D$. Then again by induction hypothesis, $l \Vdash D$.

If $k_n = l$, then by definition, $\langle k_0, \cdots, k_n \rangle \prec' \langle k_0, \cdots, k_n \rangle$, and by induction hypothesis, $\langle k_0, \cdots, k_n \rangle \Vdash' C$. Then $\langle k_0, \cdots, k_n \rangle \Vdash' D$, and again by induction hypothesis, we get $k_n \Vdash D$.

Now $\mathcal{K}'$ is a finite rooted Kripke model with root $\langle\ \rangle$ and $\mathcal{K}' \Vdash' A$ iff $\mathcal{K} \Vdash A$. $\dashv$

2.3 Kripke model theory of BA

A *Kripke model* for **BA** is a quadruple $\mathcal{K} = (K, \prec, D, \Vdash)$, like the one for **IQC**, see for example [14, Ch. 2.5.7], except that the relation $\prec$ is *not* necessarily reflexive. As for **HA**, we may attach a classical structure $\mathfrak{M}_k$ with the universe $|\mathfrak{M}_k| = D(k)$ at every node $k \in K$ such that for arbitrary $\mathbf{d} \in D(k)$ and atomic formula $A(\mathbf{d})$, $\mathfrak{M}_k \models A(\mathbf{d})$ iff $k \Vdash A(\mathbf{d})$. So the forcing relation between a node k and a sentence of the form $\forall \mathbf{x}\,(A \to B)$ is defined as

$$k \Vdash \forall \mathbf{x}\,(A \to B) \text{ if and only if for all } k' \succ k \text{ and } \mathbf{c} \in D(k'), k' \Vdash A[\mathbf{x}/\mathbf{c}] \text{ implies } k' \Vdash B[\mathbf{x}/\mathbf{c}]$$

For a formula A with free variables $\mathbf{x}$, we define

$k \Vdash A$ if and only if for all $k' \succeq k$ and all $\mathbf{c} \in D(k')$, $k' \Vdash A[\mathbf{x}/\mathbf{c}]$.

We may extend $\Vdash$ to all sequents and rules. For a sequent $A \Rightarrow B$, it is defined by

$k \Vdash A \Rightarrow B$ if and only if for all $k' \succeq k$ and $\mathbf{c} \in K$, $k' \Vdash A[\mathbf{x}/\mathbf{c}]$ implies $k' \Vdash B[\mathbf{x}/\mathbf{c}]$.

We write $\mathcal{K} \Vdash A$, A is *valid* in $\mathcal{K}$, if A is *true* in all nodes of $\mathcal{K}$, i.e. $k \Vdash A$, for all $k \in K$. For a set Γ of sequents and rules and a sequent $A \Rightarrow B$, by $\Gamma \Vdash A \Rightarrow B$ we mean that for each Kripke model $\mathcal{K}$, if $\mathcal{K} \Vdash C \Rightarrow D$ and $\mathcal{K} \Vdash R$ for all $C \Rightarrow D \in \Gamma$ and $R \in \Gamma$, then $\mathcal{K} \Vdash A \Rightarrow B$.

Theorem 2.3 (Soundness and Completeness) **BA** *is sound and strongly complete with respect to Kripke models for the language* $\mathcal{L} = \{0, S, +, \cdot\}$.

Proof. See [10]. $\dashv$

Corollary 2.4 **LA** *is sound for Kripke models with reverse well-founded frames, i.e., frames with no infinite ascending sequence of nodes.* **EBA** *is also sound for Kripke models with serial frames, i.e., frames with no end node.*

Proof. This follows from a straightforward model theoretic argument. $\dashv$

Definition 2.5 *Let* **Pos** *indicate the set of all* positive *sentences of the language of arithmetic* $\mathcal{L}$*, i.e., sentences built up of atomic formulas and* $\{\wedge, \vee, \exists\}$*.*

Lemma 2.6 *For every* $A \in$ **Pos***, and every Kripke model* $\mathcal{K} = (K, \prec, D, \Vdash)$ *of* **BA***, and every* $k \in K$*,* $k \Vdash A$ *iff* $\mathfrak{M}_k \models A$*.*

Proof. A simple induction on the complexity of A. $\dashv$

3 Some other preliminaries

In this section we collect some facts and definitions, which are used in the sequel of this paper.

Before continuing with preliminaries, let us informally sketch what we are going to do for proving our main Theorem 4.9 which states: **BPC** $\nvdash A(p_1, \ldots, p_n)$ implies **BA** $\nvdash A(A_1, \ldots, A_n)$ for some arithmetical sentences $A_1, \ldots, A_n$. From **BPC** $\nvdash A(p_1, \ldots, p_n)$, we find some finite Kripke model with tree frame $\mathcal{K} \nVdash A$. Then we construct a recursive function F (with domain of natural numbers) on $\mathcal{K}$, the function that we call it a Solovay function and is defined in the same way that first time appeared in [13]. In fact, this function is defined via recursion theorem or Gödel's diagonal lemma, and is such that $F(n)$ climbs over the frame as n becomes larger. But F reluctantly climb, e.i., $F(n+1) = F(n)$, unless $n+1$ is the code for proof (in **PA**) of "F would not remain in the node $F(n+1)$". Finally the substitutions A_i is defined to be the Σ_1-formula

$$\bigvee_{k \Vdash p_i} (\exists x F(x) = k)$$

To show that **BA** $\nvdash A(A_1, \ldots, A_n)$, we find some first-order Kripke model $\mathcal{K}_1 \nVdash A(A_1, \ldots, A_n)$. This Kripke model is constructed by a general method invented by Smoryński with the aid of Solovay function F. In the next subsection 3.1, we define the Solovay function and states some related preliminaries.

3.1 The Solovay function

Let $(K, \prec)$ be a finite tree. We fix an enumeration $K = \{k_0, \ldots, k_n\}$ such that k_0 is the root of K. Note that $k_i \neq k_j$, if $i \neq j$. To be able to define the Solovay function F in the language of arithmetic for $(K, \prec)$, we must first encode $(K, \prec)$ in the language of arithmetic, such that we can formalize the function F in **PA** and use its properties ([13]). Since $(K, \prec)$ is finite, the encoding is simple in the following sense. For $k \in K$, define its code $\overline{k}$ to be the unique numeral i (in the language of arithmetic) such that $k = k_i$. This means that $\overline{k_j} = j$ for each $0 \leq j \leq n$. After this simple encoding we may define the set K and the relation $\prec$ in the language of arithmetic as follows.

- $\overline{K}(x) := \bigvee_{k \in K} (x = \bar{k}) := \bigvee_{i=0}^{n} (x = i)$,
- $x \mathrel{\overline{\prec}} y := \bigvee_{k \prec l} (x = \bar{k} \wedge y = \bar{l})$.

In the following, we may omit over-lines if no confusion is likely.

Definition 3.1 *Let $\phi(x,y)$ be a Σ_1-formula in the language of arithmetic, and $(K,\prec)$ be a finite tree. Then*

- *$L_\phi = y$ is shorthand for $\exists x \forall z \geq x \phi(z,y)$, which informally speaking it means that the limit of a function with graph ϕ is y.*
- *For each $k \in K$, $L_\phi \succ k$ ($L_\phi \succeq k$) is shorthand for the Σ_1-sentence $\bigvee_{l \succ k} \exists x \phi(x,\bar{l})$ ($\bigvee_{l \succeq k} \exists x \phi(x,\bar{l})$),*
- *Let g be a recursive function with ϕ_g as its Σ_1-graph. The notations $L_g = y$, $L_g \succ k$ and $L_g \succeq k$ mean $L_{\phi_g} = y$, $L_{\phi_g} \succ k$ and $L_{\phi_g} \succeq k$, respectively, for any $k \in K$.*

Let $T \supseteq$ **PA** be an r.e. theory, i.e. the gödel number of its set of axioms is an r.e. set. Let $\mathsf{Pr}_{\mathbf{T}}(x)$ be a fixed Σ_1-predicate that formalizes the notion of *provability in T*. For an arithmetical sentence A, $\ulcorner A \urcorner$ is the Gödel number of A. As usual $\mathsf{Con}(T)$, the consistency of T, abbreviates $\neg \mathsf{Pr}_T(\ulcorner 0 = 1 \urcorner)$. In the following Theorem, $\mathbb{N}$ is the standard model of **PA**.

Theorem 3.2 *For any finite tree $(K,\prec)$ with the root k_0, there exists a Σ_1-formula $\varphi(x,y)$ in the language of arithmetic satisfying the following conditions:*

1. **PA** $\vdash \forall x \exists ! y \varphi(x,y)$, *i.e. $\varphi(x,y)$ is the graph of a provably total recursive function. Hence we can freely use $F(x) = y$ instead of $\varphi(x,y)$.*
2. **PA** $\vdash x \leq y \to F(x) \preceq F(y)$.
3. *For $k,l \in K$ with $k \prec l$,* **PA** $+ L_F = \bar{k} \vdash \neg \mathsf{Pr}_{PA}(\ulcorner \neg L_F = \bar{l} \urcorner)$, *i.e.,* **PA** $+ L_F = \bar{k}$ *implies consistency of* **PA** $+ L_F = \bar{l}$.
4. $\mathbb{N} \models L_F = \bar{k_0}$.

Proof. All the above assertions are from [11, Page 136-140], with some minor differences which we explain below.

- The language of arithmetic in [11], is $\mathcal{L}$ augmented with function symbols for each primitive recursive function. Let us denote this language by $\mathcal{L}^+$. But we know that all primitive recursive functions are provably total in **PA** and moreover, are Δ_0-definable in **PA**. This implies that the class of Σ_1-formulas is the same in both languages (up to equivalence in **PA**), i.e. each Σ_1-formula in $\mathcal{L}^+$ is equivalence to a Σ_1-formula in $\mathcal{L}$. Also **PA**$^+$ is $\mathcal{L}$-conservative over **PA**.
- The theory which is assumed in [11] is **PRA** in the language $\mathcal{L}^+$. This theory is weaker than **PA**$^+$ (the same theory **PA** formalized in the extended language $\mathcal{L}^+$), and hence the above assertions are weaker in this sense.

$\dashv$

It should be mentioned that we are not going to add a new function symbol F to the language. We only make the convention that $F(x) = y$ means $\varphi(x,y)$, and for example, for an arbitrary formula $B(x)$, we replace $B(F(x))$ with $\exists y (F(x) = y \wedge B(y))$ and so on. For more details see [8, Sec. 4.2]. R. Solovay in [13] used this function to prove the completeness of arithmetical interpretations for the provability logic of **PA**. We make use of this function to prove the de Jongh property for **BA**. It is worth mentioning that the method used by Smoryński [12, Sec. 5.6.13] to provide Σ-substitutions for the de Jongh property for **HA**, works for our purpose as well. We prefer here to use rather convenient, the Solovay function.

3.2 The MRDP-Theorem

The celebrated Theorem of Matiyasevich, Robinson, Davis and Putnam, known as the **MRDP**-Theorem, states that every recursively enumerable set is Diophantine, i.e. it is the set of solutions of a multivariate polynomial with integer coefficients [9]. We need the following equivalent form of this Theorem in our proof of the main Theorem of this paper, i.e., Theorem 4.9, in which we would like to get rid of the universal bounded quantifiers in Kripke model theory of **BA**.

Theorem 3.3 *For each Σ_1-formula A, there exists some polynomials (some terms in the language of arithmetic) t_1 and t_2 such that* $\mathbf{PA} \vdash A \leftrightarrow \exists \mathbf{x}(t_1(\mathbf{x}) = t_2(\mathbf{x}))$.

Proof. In [6], it is proven that there exists some quantifier-free formula $B(\mathbf{x})$ such that $\mathbf{PA} \vdash A \leftrightarrow \exists \mathbf{x} B$. Now by the following simple observations one can show the desired result.

- B can be written in conjunctive normal form, i.e., $\mathbf{PA} \Vdash B \leftrightarrow \bigwedge_i \bigvee_j B_{i,j}$, in which $B_{i,j}$ are atomic or negation of atomic formulas,
- Each atomic formula is of the form $t = s$, in which t and s are terms,
- $\mathbf{PA} \vdash \neg(t = s) \leftrightarrow \exists x((t + x + 1 = s) \vee (s + x + 1 = t))$,
- $\mathbf{PA} \vdash (t_1 = s_1 \wedge t_2 = s_2) \leftrightarrow (t_1^2 + t_2^2 + s_1^2 + s_2^2 = 2t_1 s_1 + 2t_2 s_2)$,
- $\mathbf{PA} \vdash (t_1 = s_1 \vee t_2 = s_2) \leftrightarrow (t_1 t_2 + s_1 s_2 = t_1 s_2 + t_2 s_1)$.

$\dashv$

3.3 Interpretability

Several notions of interpretability have been studied in the literature. The one that we are going to consider is defined in [7, p.491] or [3, 15], to which we refer the reader for a precise definition.

Assume that T, S are two arithmetical theories. *An interpretation I of S in T* is a function $(\cdot)^I$ on arithmetical formulas such that $T \vdash A^I$ if and only if $S \vdash A$, for any formula A in the language of arithmetic. Any interpretation $(.)^I$ (of our interest) commutes with propositional connectives and can be presented by four formulas: $\delta(x)$, $A(x, y, z)$, $M(x, y, z)$, $Suc(x, y)$, representing the universe, interpretation of (graph of) addition, multiplication and successor functions respectively. For example, the interpretation of $x + y = z$ is defined as $A(x, y, z)$ and $\exists x A$ is interpreted as $\exists x(\delta(x) \wedge A^I)$, in which A^I is the interpretation of A. Both $=$ and 0 are interpreted as themselves. We use the shorthand notation $T \rhd_I S$ to mean that I is an interpretation of S in T. Then $T \rhd S$ means that there exists an interpretation

I such that $T \rhd_I S$. Assume $T \rhd_I S$. Then from any $\mathfrak{M} \models T$, we can find a model $\mathfrak{M}^I$, such that $\mathfrak{M}^I \models A$ iff $\mathfrak{M} \models A^I$. The model $\mathfrak{M}^I$ is defined as follows. The universe of this model is the set of all $m \in |\mathfrak{M}|$ that $\mathfrak{M} \models \delta(\overline{m})$. All the relations and functions are defined on this universe to be as the formulas $\delta(x), A(x, y, z), M(x, y, z), Suc(x, y)$ are in $\mathfrak{M}$. For two arithmetical models $\mathfrak{M}$ and $\mathfrak{N}$, we say that $\mathfrak{M} \rhd \mathfrak{N}$, iff there exists some interpretation I such that $\mathfrak{N}$ is isomorphic to $\mathfrak{M}^I$. For more details of definitions we refer the reader to [15]. We collect all we need about interpretability in the following Theorem.

Theorem 3.4 *1. For any r.e. theory T, $\mathbf{PA} + \mathsf{Con}(T) \rhd T$.*

2. Let T, S be two arithmetical theories extending **PA***, and $T \rhd S$. Then for any model $\mathfrak{M} \models T$, there exists an end-extension $\mathfrak{N}$ of $\mathfrak{M}$ such that $\mathfrak{N} \models S$ and $\mathfrak{M} \rhd \mathfrak{N}$.*

Proof. For the proof of the first part, see [7, Th. 6.2], and for the second part, see [3, Th. 2.12]. $\dashv$

4 The de Jongh theorem for BPC

We will first show that Smoryński's general method of providing finite Kripke models of **HA** [12, Sec. 5.6.1], could be applied to **BA** as well. From that we prove the de Jongh property for **BA** and its extensions.

Assume a classical structure $\mathfrak{M}$ for the language of arithmetic, $\mathcal{L}$. The extended language $\mathcal{L}_{\mathfrak{M}}$ is defined to be the language $\mathcal{L}$, augmented with a new constant symbol $\bar{d}$ for each $d \in |\mathfrak{M}|$. In the rest of this paper, we may omit over-lines when no confusion is likely.

We recall that $\mathfrak{M}$ is a *weak substructure* of $\mathfrak{N}$, if the universe of $\mathfrak{M}$ ($|\mathfrak{M}|$) is a subset of the universe of $\mathfrak{N}$ and for all atomic formula $A \in \mathcal{L}_{\mathfrak{M}}$, $\mathfrak{M} \models A$ implies $\mathfrak{N} \models A$.

Smoryński [12, Sec. 5.6] defined a very useful operation $(.)^*$ on first-order Kripke models of **HA**. The following definitions of $\mathcal{K}^\circ_{\mathfrak{M}}$ and $\mathcal{K}^\bullet_{\mathfrak{M}}$ are similar to Smoryński's operation for **BA**.

Definition 4.1 *Suppose that $\mathcal{K} = (K, \prec, D, \Vdash)$ is a first-order Kripke model for the language of arithmetic, and $\mathfrak{M} \models$* **PA** *is a classical model such that for each $k \in K$, $\mathfrak{M}$ is a* weak substructure *(definition will come in the sequel) of $\mathfrak{M}_k$. We define*

- $\mathcal{K}^\circ_{\mathfrak{M}} := (K^\circ, \prec^\circ, D^\circ, \Vdash^\circ)$*, and*
- $\mathcal{K}^\bullet_{\mathfrak{M}} := (K^\bullet, \prec^\bullet, D^\bullet, \Vdash^\bullet)$*,*

to be the first-order Kripke model obtained from $\mathcal{K}$, by adding a new irreflexive and reflexive structure $\mathfrak{M}$ in beneath of the others, respectively, i.e.

- $K^\circ := K^\bullet := K \cup \{k_0\}$*, in which k_0 is a fresh node,*
- $\prec^\circ := \prec \cup (\{k_0\} \times K)$ *and* $\prec^\bullet := \prec^\circ \cup \{(k_0, k_0)\}$*,*
- *for any $k \in K$, $D^\circ(k) := D^\bullet(k) := D(k)$, and $D^\circ(k_0) := D^\bullet(k_0) := |\mathfrak{M}|$,*

- *and finally, for each $k \in K$ and atomic A, $k \Vdash^\circ A$, $k \Vdash^\bullet A$ are defined as $k \Vdash A$, and $k_0 \Vdash^\circ A$, $k_0 \Vdash^\bullet A$ are defined as $\mathfrak{M} \models A$.*

We recall the classical notion of *initial segment* from literature [8, Sec. 2.2]. Let $\mathfrak{M}$ and $\mathfrak{N}$ be two classical models in a language with an ordering relation $\leq$, such that $\mathfrak{M}$ is substructure of $\mathfrak{N}$. We say that $\mathfrak{M}$ is an *initial segment* of $\mathfrak{N}$ (or $\mathfrak{N}$ is an *end-extension* of $\mathfrak{M}$) iff for all $m \in |\mathfrak{M}|$, $\mathfrak{N} \models n \leq m$, for some $n \in |\mathfrak{N}|$ implies $n \in |\mathfrak{M}|$. We emphasize that $|\mathfrak{M}| \subseteq |\mathfrak{N}|$ is a necessary condition for $\mathfrak{M}$ to be an initial segment of $\mathfrak{N}$.

The following definition is similar to *definability of first-order Kripke models of* **HA** *in nonstandard models of* **PA** [12, Sec. 5.6] for **BA**.

Definition 4.2 *We say that a classical model $\mathfrak{M} \models$ **PA**,* interprets *a first-order finite Kripke model $\mathcal{K} = (K, \prec, D, \Vdash)$, indicated by $\mathfrak{M} \rhd \mathcal{K}$, iff the following conditions hold:*

- *$\mathfrak{M}$ is an initial segment of the classical structure $\mathfrak{M}_k$ attached to node $k \in K$, and moreover, $\mathfrak{M}_k \models$ **PA**, for any $k \in K$.*
- *There exist functions $f : \bigcup_{k \in K} D(k) \longmapsto |\mathfrak{M}|$ and $g : K \longmapsto |\mathfrak{M}|$, such that $f(m) = m$, for all $m \in |\mathfrak{M}|$.*
- *There exist arithmetical formulas $R_\prec(x, y)$ and $D(x, y)$, with free variables as indicated, such that*
  - *For arbitrary $k, l \in |\mathfrak{M}|$, $\mathfrak{M} \models R_\prec(k, l)$ iff there are $k' \prec l' \in K$ such that $k = g(k')$ and $l = g(l')$.*
  - *For arbitrary $k, a \in |\mathfrak{M}|$, $\mathfrak{M} \models D(k, a)$ iff there are $k' \in K$ and $a' \in D(k)$, such that $g(k') = k$ and $f(a') = a$.*
- *For every $k \in K$ and every atomic formula $A(x_1, \ldots, x_m)$, there exists a formula $A^{I_k}(x_1, \ldots, x_m)$, such that for any $d_1, \ldots, d_m \in D(k)$,*

$$\mathfrak{M}_k \models A(d_1, \ldots, d_n) \text{ iff } \mathfrak{M} \models A^{I_k}(f(d_1), \ldots, f(d_m)).$$

Let $\mathcal{F} := \{\mathcal{K}_i\}_{i \in I}$ be a family of first-order Kripke models, i.e., for each $i \in I$, $\mathcal{K}_i = (K_i, \prec_i, D_i, \Vdash_i)$. The *disjoint union* of $\mathcal{F}$, indicated by $\Sigma\mathcal{F}$ [12], is defined as the Kripke model built up from putting all the Kripke models in $\mathcal{F}$ altogether. Formally, we define $\Sigma\mathcal{F} := (K, \prec, D, \Vdash)$, in which

$$K := \bigcup_{i \in I} \{i\} \times K_i \quad , \quad \prec := \bigcup_{i \in I} \prec'_i \quad , \quad D := \bigcup_{i \in I} D'_i \quad , \quad \Vdash := \bigcup_{i \in I} \Vdash'_i,$$

where $\prec'_i := \{((i, a), (i, b)) \mid (a, b) \in \prec_i\}$, D'_i is a function with domain $\{i\} \times K_i$ such that $D'_i(i, k) := D_i(k)$, and $\Vdash'_i := \{((i, a), p) \mid (a, p) \in \Vdash_i\}$.

The following lemma is similar to Lemma 5.6.4 in [12].

Lemma 4.3 *Let $\mathcal{F} = \{\mathcal{K}_i\}_{i \in I}$ be a finite family of finite first-order Kripke models and $\mathfrak{M}$ be a classical model of* **PA** *such that, for each $i \in I$, $\mathfrak{M} \rhd \mathcal{K}_i$ and $\mathcal{K}_i = (K_i, \prec_i, D_i, \Vdash_i)$ and for any $i \neq j$ and $k \in K_i, k' \in K_j$, we have $D_i(k) \cap D_j(k') = |\mathfrak{M}|$. Then $\mathfrak{M} \rhd \Sigma\mathcal{F}$.*

Proof. Let $I = \{1, 2, \ldots, n\}$. The assumption $\mathfrak{M} \rhd \mathcal{K}_i$, for $1 \leq i \leq n$, gives us f_i, g_i, $R_{\prec_i}(x, y)$, $D_i(x, y)$, as introduced in Definition 4.2. Now we define $f, g, R_{\prec}(x, y)$ and $D(x, y)$ for the new model $\Sigma\mathcal{F}$ as follows.

- $f : \bigcup_{i=1}^{i=n} \bigcup_{k \in K_i} D_i(k) \longmapsto |\mathfrak{M}|$, by $f(d) = f_i(d)$, where $d \in \bigcup_{k \in K_i} D_i(k)$. Note that the function defined in this way is well defined, since $D_i(k) \cap D_j(k') = |\mathfrak{M}|$, for each $i \neq j$, and $f_i(d) = d$ for each $d \in |\mathfrak{M}|$.
- $g : \bigcup_{i=1}^{i=n} \{i\} \times K_i \longmapsto |\mathfrak{M}|$, by $g(i, k) := \langle i, g_i(k) \rangle$. Note that $\langle -, - \rangle$ is a one-to-one pairing function with inverses, satisfying $l(\langle x, y \rangle) = x$ and $r(\langle x, y \rangle) = y$.
- $R_{\prec}(x, y) := \bigvee_i [l(x) = l(y) = i \wedge R_{\prec_i}(r(x), r(y))]$.
- $D(x, y) := \bigvee_i [l(x) = i \wedge D_i(r(x), y)]$.

Finally, assume that for each $k \in K_i$, the translation $(\cdot)^{I_{(i,k)}}$ be $(\cdot)^{I_k}$, as is provided by $\mathfrak{M} \rhd \mathcal{K}_i$. Now it is straightforward to show that $\mathfrak{M} \rhd \Sigma\mathcal{F}$. $\dashv$

Lemma 4.4 *Suppose that $\mathcal{K}$ is a finite first-order Kripke model and $\mathfrak{M}$ is a classical model of* **PA** *such that $\mathfrak{M} \rhd \mathcal{K}$. Then $\mathfrak{M} \rhd \mathcal{K}^{\circ}_{\mathfrak{M}}$ and $\mathfrak{M} \rhd \mathcal{K}^{\bullet}_{\mathfrak{M}}$.*

Proof. It is straightforward. $\dashv$

The following lemma is analogue of Lemma 5.6.3 in [12].

Lemma 4.5 *Suppose that $\mathcal{K} = (K, \prec, D, \Vdash)$ and $\mathfrak{M} \rhd \mathcal{K}$. Let the functions f, g be given as in Definition 4.2. Then for each sequent $A(x_1, \ldots, x_n)$, there exists a formula $F_A(x_0, x_1, \ldots, x_n)$ in the language $\mathcal{L}_{\mathfrak{M}}$ such that for each $k \in K$ and each $d_1, \ldots, d_n \in D(k)$,*

$$k \Vdash A(d_1, \ldots, d_n) \text{ iff } \mathfrak{M} \models F_A(g(k), f(d_1), \ldots, f(d_n)).$$

Proof. Assume the arithmetical formulas $R_{\prec}(x, y), D(x, y)$ and $A^{I_k}(x_1, \cdots, x_n)$ are as introduced in Definition 4.2. Then we define, by induction on A, the formula F_A, as follows. Let us indicate an n-tuple of variables $x_1, \cdots, x_n$ by $\mathbf{x}$.

- For any atomic formula $A(\mathbf{x})$, take $F_A(x_0, \mathbf{x}) := \bigvee_{k \in K} (x_0 = g(k) \wedge A^{I_k}(\mathbf{x}))$.
- For $A(\mathbf{x}) = A_1(\mathbf{x}) \vee A_2(\mathbf{x})$, let $F_A(x_0, \mathbf{x}) := F_{A_1}(x_0, \mathbf{x}) \vee F_{A_2}(x_0, \mathbf{x})$. Similarly for $\wedge$.
- For $A(\mathbf{x}) = \exists y B(y, \mathbf{x})$, take $F_A(x_0, \mathbf{x}) := \exists y (D(x_0, y) \wedge F_B(x_0, y, \mathbf{x}))$.
- For $A(\mathbf{x}) = \forall y_1 y_2 \ldots y_n (B_1(\mathbf{x}, y_1, \ldots, y_n) \to B_2(\mathbf{x}, y_1, \ldots, y_n))$, let

$$F_A(x_0, \mathbf{x}) := \forall z \forall y_1 \ldots y_n ([R_{\prec}(x_0, z) \wedge \bigwedge_{1 \leq i \leq n} D(z, y_i) \wedge F_{B_1}(z, \mathbf{x}, y_1, \ldots y_n)] \to F_{B_2}(z, \mathbf{x}, y_1, \ldots, y_n))$$

- Finally, if A is a sequent, i.e., $A_1(\mathbf{x}) \Rightarrow A_2(\mathbf{x})$, take

$$F_A(x_0, \mathbf{x}) := \forall z [((R_{\prec}(x_0, z) \vee x_0 = z) \wedge F_{A_1}(z, \mathbf{x})) \to F_{A_2}(z, \mathbf{x})].$$

Now it is routine to show, by induction, the desired property for F_A. $\dashv$

Lemma 4.6 *Suppose that $\mathcal{K}$ is a finite Kripke model of* **BA**, *$\mathfrak{M}$ is a classical model of* **PA** *and $\mathfrak{M} \rhd \mathcal{K}$. Then $\mathcal{K}^{\circ}_{\mathfrak{M}}$ and $\mathcal{K}^{\bullet}_{\mathfrak{M}}$ are also models of* **BA**.

Proof. First observe that by Lemma 4.4, $\mathfrak{M} \rhd \mathcal{K}^{\circ}_{\mathfrak{M}}$ and $\mathfrak{M} \rhd \mathcal{K}^{\bullet}_{\mathfrak{M}}$. Let f°, g° and $f^{\bullet}, g^{\bullet}$ be the functions given by $\mathfrak{M} \rhd \mathcal{K}^{\circ}_{\mathfrak{M}}$ and $\mathfrak{M} \rhd \mathcal{K}^{\bullet}_{\mathfrak{M}}$, respectively. It is clear from the definitions that $f^{\circ} = f^{\bullet}$ and $g^{\circ} = g^{\bullet}$, hence we can omit superscripts. For arbitrary sequent $A(\mathbf{x})$, by Lemma 4.5, we can also find $F^{\circ}_A(x_0, \mathbf{x})$ and $F^{\bullet}_A(x_0, \mathbf{x})$ with the mentioned properties.

All the axioms and rules of **BA** are trivially valid in both $\mathcal{K}^{\circ}_{\mathfrak{M}}$ and $\mathcal{K}^{\bullet}_{\mathfrak{M}}$, except the induction axiom and rule, which need more caution. It is clear that the induction axiom and rule are forced at any node other than the root k_0 in both models $\mathcal{K}^{\circ}_{\mathfrak{M}}$ and $\mathcal{K}^{\bullet}_{\mathfrak{M}}$. So we consider only the root k_0.

1. *Induction axiom in $\mathcal{K}^{\circ}_{\mathfrak{M}}$.* Suppose that $k_0 \Vdash^{\circ} \forall \mathbf{y} x(A \to A[x/S(x)])$. We should show $k_0 \Vdash^{\circ} \forall \mathbf{y} x(A[x/0] \to A)$. By definition of forcing and $k_0 \Vdash^{\circ} \forall \mathbf{y} x(A \to A[x/S(x)])$, we have $\mathcal{K} \Vdash A \Rightarrow A[x/S(x)]$. Since the induction rule is valid in $\mathcal{K}$, we get $\mathcal{K} \Vdash A[x/0] \Rightarrow A$. Then definition of forcing and irreflexivity of k_0 implies that $k_0 \Vdash^{\circ} \forall \mathbf{y} x(A[x/0] \to A)$, as desired.

2. *Induction axiom in $\mathcal{K}^{\bullet}_{\mathfrak{M}}$.* Suppose that $k_0 \Vdash^{\bullet} \forall \mathbf{y} x(A \to A[x/S(x)])$. By validity of the induction rule in $\mathcal{K}$ and reflexivity of k_0, it is enough to show that for arbitrary $m, \mathbf{a} \in |\mathfrak{M}|$, $k_0 \Vdash^{\bullet} A(0, \mathbf{a})$ implies $k_0 \Vdash^{\bullet} A(m, \mathbf{a})$. Fix some $\mathbf{a} \in |\mathfrak{M}|$ and let $k_0 \Vdash^{\bullet} A(0, \mathbf{a})$. Since k_0 is a reflexive node, for arbitrary $m \in |\mathfrak{M}|$, $k_0 \Vdash A(m, \mathbf{a})$ implies $k_0 \Vdash A(S(m), \mathbf{a})$. Now Lemma 4.5 implies

$$\mathfrak{M} \models F^{\bullet}_A(g(k_0), 0, \mathbf{a}) \wedge \forall x [F^{\bullet}_A(g(k_0), x, \mathbf{a}) \to F^{\bullet}_A(g(k_0), S(x), \mathbf{a})].$$

   Then, by validity of induction in $\mathfrak{M}$, we have $\mathfrak{M} \models \forall x F_A(g(k_0), x, \mathbf{a})$. Hence by Lemma 4.5, $k_0 \Vdash^{\bullet} A(m, \mathbf{a})$, for arbitrary $m \in |\mathfrak{M}|$, as desired. Note that in the above argument, we used the fact $f(m) = m$, for any $m \in |\mathfrak{M}|$.

3. The other cases, i.e. the induction rule in $\mathcal{K}^{\circ}$ and $\mathcal{K}^{\bullet}$, can be treated similarly.

$\dashv$

Definition 4.7 *An I-frame is a triple $\mathcal{I} = (K, \prec, T)$, that has the following properties:*

- *$(K, \prec)$ is a tree.*
- *T is a function on K and for all $k \in K$, $T(k)$ is a consistent theory containing* **PA**.
- *For all $k \prec l$, we have $T(k) \rhd T(l)$.*

Theorem 4.8 *For any I-frame $\mathcal{I} = (K, \prec, T)$ with rooted finite tree frame, we can find a Kripke model $\mathcal{K}$ with the same frame $(K, \prec)$, such that $\mathcal{K} \Vdash$* **BA** *and $\mathfrak{M}_k \models T(k)$, for each $k \in K$.*

Proof. We prove (a strengthening of the statement of theorem) by induction on the height of the rooted tree $(K, \prec)$ that, for any I-frame $(K, \prec, T)$ and arbitrary set $X \supseteq |\mathfrak{M}_{k_0}|$, if k_0 is the root of $(K, \prec)$ and $\mathfrak{M}_{k_0}$ is a classical model such that $\mathfrak{M}_{k_0} \models T(k_0)$, then there exists a first-order Kripke model $\mathcal{K} = (K, \prec, D, \Vdash)$, such that $\mathfrak{M}_{k_0}$ is the classical structure attached to the node k_0, and for any $k \in K$, we have $\mathfrak{M}_k \models T(k)$, $D(k) \cap X = |\mathfrak{M}_{k_0}|$, $\mathfrak{M}_{k_0} \rhd \mathcal{K}$ and moreover, $\mathcal{K} \Vdash \mathbf{BA}$.

Suppose that the claim is true for all frames with height lower than h. Let $\mathcal{I} = (K, \prec, T)$ be an I-frame with rooted finite tree frame and height h. Let also that $k_0 \in K$ be the root of $(K, \prec)$ and $\mathfrak{M}_{k_0} \models T(k_0)$ be a classical model and also assume that X is an arbitrary set containing $|\mathfrak{M}_{k_0}|$. Let $S := \{k_i | 1 \leq i \leq n\}$ be the set of all immediate successors of k_0, i.e. $k_i \succ k_0, k_0 \neq k_i$, and for each $l \in K$ with $k_0 \preceq l \preceq k_i$, then either $l = k_0$ or $l = k_i$. We define, by induction on i, the first order Kripke models $\mathcal{K}_i$ as follows.

Suppose that for each $j < i$, we have already defined $\mathcal{K}_j = (K_j, \prec_j, D_j, \Vdash_j)$. Since $T(k_0) \rhd T(k_i)$, Theorem 3.4 implies that there exists a classical model $\mathfrak{M}_{k_i} \models T(k_i)$, such that $\mathfrak{M}_{k_0} \rhd \mathfrak{M}_{k_i}$ and $\mathfrak{M}_{k_i}$ is an end-extension of $\mathfrak{M}_{k_0}$. We may take $\mathfrak{M}_{k_i}$ such that $|\mathfrak{M}_{k_i}| \cap X_i = |\mathfrak{M}_{k_0}|$, in which $X_i = X \cup \bigcup_{j<i, k \in K_j} D_j(k)$. That is plausible, since we can replace $\mathfrak{M}_{k_i}$ with a suitable isomorphic model $\mathfrak{M}'_{k_i}$ that fulfills the required property. Now apply the induction hypothesis to the I-frame $\mathcal{I}_{k_i}$, the set $X_i \cup |\mathfrak{M}_{k_i}|$ and $\mathfrak{M}_{k_i}$, in which $\mathcal{I}_k$ is the restriction of $\mathcal{I}$ to the nodes above or equal to k. This will provide a first-order Kripke model $\mathcal{K}_i$ with the mentioned properties

By composition of the interpretations $\mathfrak{M}_{k_0} \rhd \mathfrak{M}_{k_i}$ and $\mathfrak{M}_{k_i} \rhd \mathcal{K}_i$, one can simply extract an interpretation $\mathfrak{M}_{k_0} \rhd \mathcal{K}_i$. Now define $\mathcal{F} := \{\mathcal{K}_i \mid 1 \leq i \leq n\}$, and finally,

$$\mathcal{K} := \begin{cases} (\Sigma\mathcal{F})^{\bullet}_{\mathfrak{M}_{k_0}} & \text{if } k_0 \text{ is reflexive,} \\ (\Sigma\mathcal{F})^{\circ}_{\mathfrak{M}_{k_0}} & \text{if } k_0 \text{ is irreflexive.} \end{cases}$$

Now Lemma 4.3 guarantees $\mathfrak{M}_{k_0} \rhd \Sigma\mathcal{F}$, Lemma 4.4 implies $\mathfrak{M}_{k_0} \rhd \mathcal{K}$ and Lemma 4.6 implies $\mathcal{K} \Vdash \mathbf{BA}$. It is straightforward to check that the other required properties mentioned in induction hypothesis hold as well. $\dashv$

Theorem 4.9 *For any propositional sequent $A(p_1, \ldots, p_n)$,* $\mathbf{BPC} \vdash A(p_1, \ldots, p_n)$ *iff* $\mathbf{BA} \vdash A(A_1, \ldots, A_n)$ *for all arithmetical sentences $A_1, \ldots, A_n$.*

Proof. Left to right derivation can be shown by a straightforward induction on the length of the proof $\mathbf{BPC} \vdash A(p_1, \ldots, p_n)$.

For the other side, assume that $\mathbf{BPC} \nvdash A(p_1, \ldots, p_n)$. We will find some Σ_1-sentences $A_1, \ldots, A_n$, such that $\mathbf{BA} \nvdash A(A_1, \ldots, A_n)$. By Theorem 2.2, there is a finite rooted tree Kripke model $\mathcal{K}$ with root k_0, such that $\mathcal{K} \nVdash A(p_1, \ldots, p_n)$. Let F be the Solovay function on the frame of $\mathcal{K}$, as given by Theorem 3.2. Now define $\mathcal{I} := (K, \prec, T)$, in which $T(k) := \mathbf{PA} + L_F = \bar{k}$ for each $k \in K$. Theorem 3.2 implies that for any $k \prec l \in K$, we have $T(k) \vdash \mathsf{Con}(T(l))$, and so by Theorem 3.4, $T_k \rhd T_l$. By Theorem 3.2, $\mathbb{N} \models L = k_0$, hence $T(k_0)$ is consistent. Since $T(k_0) \rhd T(k)$, for any $k \in K$, by Theorem 3.4, $T(k)$ is consistent, for any $k \in K$. Hence $\mathcal{I}$ is an I-frame. Then Theorem 4.8 implies that there exists a first-order Kripke model $\mathcal{K}_1 := (K, \prec, D, \Vdash_{\mathcal{K}_1})$ with the same frame $(K, \prec)$, such that for each $k \in K$, $\mathfrak{M}_k \models T(k)$. For each i we define $B_i := \bigvee_{k \Vdash p_i} \exists x F(x) = \bar{k}$, which is a

Σ_1-sentence in the language of **PA**. By **MRDP**-Theorem 3.3, there exists some sentence $A_i \in$ **Pos** such that **PA** $\vdash A_i \leftrightarrow B_i$. Now by induction on the complexity of B, one can prove that for every $k \in K$,

$$k \Vdash_{\mathcal{K}} B(p_1, \ldots, p_n) \text{ iff } k \Vdash_{\mathcal{K}_1} B(A_1, \ldots, A_n).$$

The only case that we should treat is the atomic case. The other cases follow by the inductive definition of "$\Vdash$" in Kripke models.

Suppose that $B = p_i$ is atomic and $k \Vdash_{\mathcal{K}} p_i$. Then the definition of $L_F = k$ implies that **PA** $\vdash L_F = \bar{k} \rightarrow \exists x F(x) = \bar{k}$, hence $T(k) \vdash B_i$, which implies $T(k) \vdash A_i$. Now by Theorem 4.8, $\mathfrak{M}_k \models T(k)$, and so by Lemma 2.6, $k \Vdash_{\mathcal{K}_1} A_i$.

For the other side, assume that $k \nVdash p_i$. Then for any $l \in K$ such that $l \Vdash p_i$, we have $l \not\preccurlyeq k$. Hence by Theorem 3.2, **PA** $\vdash L_F = \bar{k} \rightarrow \neg\exists x F(x) = \bar{l}$, for any $l \not\preccurlyeq k$. So **PA** $\vdash L_F = \bar{k} \rightarrow \neg B_i$, which implies $T(k) \vdash \neg A_i$. Hence by Lemma 2.6, $k \nVdash_{\mathcal{K}_1} A_i$, as desired. So $k_0 \nVdash A(A_1, \ldots, A_n)$, i.e. $\mathcal{K}_1 \nVdash A(A_1, \ldots A_n)$. Now Theorem 2.3 implies **BA** $\nvdash A(A_1, \ldots, A_n)$, as desired. $\dashv$

Remark 4.10 *As the above proof shows, we may strengthen the completeness part of the above Theorem as follows. For each propositional sequent $A(p_1, \ldots, p_n)$, if for any Σ_1-sentences $A_1, \ldots, A_n$, we have* **BA** $\vdash A(A_1, \ldots, A_n)$*, then* **BPC** $\vdash A(p_1, \ldots, p_n)$

A closer look at the above proof for de Jongh property for **BA** reveals that the same proof works for **HA**, **LA** and **EBA** as well.

Remark 4.11 *For any propositional sequent $A(p_1, \ldots, p_n)$,*

1. **IPC** $\vdash A(p_1, \ldots, p_n)$ *iff* **HA** $\vdash A(B_1, \ldots, B_n)$*, for all arithmetical sentences $B_1, \ldots, B_n$.*

2. **FPC** $\vdash A(p_1, \ldots, p_n)$ *iff* **LA** $\vdash A(B_1, \ldots, B_n)$*, for all arithmetical sentences $B_1, \ldots, B_n$.*

3. **EBPC** $\vdash A(p_1, \ldots, p_n)$ *iff* **EBA** $\vdash A(B_1, \ldots, B_n)$*, for all arithmetical sentences $B_1, \ldots, B_n$.*

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