The Σ_1-Provability Logic of HA Revisited

Mojtaba Mojtahedi*
University of Tehran, College of science
Department of Mathematics, Statistics and Computer Sciences

November 20, 2020

Dedicated to Dick de Jongh on the occasion of his 80th birthday.

Abstract

The Σ_1-provability logic of Peano Arithmetic PA, is characterized [Visser, 1982] as $\mathsf{GLC_a}$, the Gödel-Löb logic GL plus the completeness principle for atomic variables. Also the Σ_1-provability logic of the Heyting Arithmetic HA, is characterized [Ardeshir and Mojtahedi, 2018] as iH'_σ (for definition of iH'_σ, see section 7). In this paper, we find some translation $(.)^{\boxed{h}}$, which embeds iH'_σ in $\mathsf{iGLC_a}$, the intuitionistic counterpart of $\mathsf{GLC_a}$.

Contents

1 Preface **2**

2 Introduction **2**

3 Basic definitions **2**

4 NNIL propositions **3**
4.1 The NNIL-algorithm 4
4.2 The binary relation $\blacktriangleright$ 5
4.3 $\mathsf{NNIL}^\Box$ propositions 6

5 Leivant's translation **7**

6 Gödel's translation and Heyting's Normal Form **8**

7 Axiom schemas and theories **11**

8 Embedding of iH'_σ in $\mathsf{iGLC_a}$ **13**

*http://mmojtahedi.ir

1 Preface

The first arithmetical completeness result, is due to Dick de Jongh in 1969 where he proves in an unpublished paper that the intuitionistic logic is complete for arithmetical interpretations in the Heyting's Arithmetic HA. This result appears even before the Solovay's arithmetical completeness results for the provability logic of PA [Solovay, 1976]. de Jongh's proof uses an ingenious combination of Kleene's realizability with Kripke semantics. An extended abstract of this result appears in [de Jongh, 1970]. For a short survey on *de Jongh's theorem*, we refer the reader to [de Jongh et al., 2011]. The author of this paper, first entered the fascinating world of the arithmetical completenesses, through *de Jongh's theorem*, where together with Mohammad Ardeshir, author of this paper proved de Jongh theorem for the Basic Arithmetic [Ardeshir and Mojtahedi, 2014] and also for $\mathsf{HA}+\neg\mathsf{Con}(\mathsf{HA})$ [Ardeshir and Mojtahedi, 2018].

2 Introduction

There are at least three reliable surveys on the subject of provability logic [Japardize and de Jongh, 1997; Artemov and Beklemishev, 2004; Beklemishev and Visser, 2006]. Here for the sake of completeness we introduce very shortly the subject and try to cover the recent results related to intuitionistic provability logics. Historically, the first provability interpretation for modal operator $\Box$, considered by Gödel in [Gödel, 1933]. Then the very natural question raised: what remains of a first-order theory T in the modal language, when we interpret $\Box$ as provability in T? The answer to this question for strong theories is stable: all strong enough theories like PA and ZF share the same provability logic GL [Solovay, 1976]. Then [Visser, 1982] characterizes the Σ_1-provability logic of PA as $\mathsf{GLC_a} := \mathsf{GL}$ plus the completeness principle $p \to \Box p$ for atomic variables p. [Ardeshir and Mojtahedi, 2018; Visser and Zoethout, 2019] characterize the Σ_1-provability logic of HA (the intuitionistic counterpart of PA) as iH'_σ. For definition of iH'_σ, see section 7. In this paper, we show that iH'_σ is embeddable in $\mathsf{iGLC_a}$. Before we explain more about this embedding, let us see an example of such embeddings. The Gödel's negative translation $A^\neg$, puts double negation behind every subformula of A. It is well-known that this translation, embeds classical logic into the intuitionistic logic. In this paper (theorem 8.10), we define a translation $(.)^{\boxed{h}}$ such that

- $\mathsf{iH}'_\sigma \vdash A \leftrightarrow A^{\boxed{h}}$,
- $\mathsf{iH}'_\sigma \vdash A$ iff $\mathsf{iGLC_a} \vdash A^{\boxed{h}}$,

This result is interesting because, it shows that while the Σ_1-provability logic of HA has some additional axioms over $\mathsf{iGLC_a}$ (see section 7), it can be embedded in $\mathsf{iGLC_a}$, which is the intuitionistic counterpart of the Σ_1-provability logic of PA.

3 Basic definitions

The propositional modal language is $\mathcal{L}_\Box := \{\to, \vee, \wedge, \bot, \Box\}$. The propositional non-modal language is indicated by $\mathcal{L}_0 := \{\to, \vee, \wedge, \bot\}$. We use $\boxdot A$ as a shorthand for $A \wedge \Box A$. IPC is the intuitionistic propositional non-modal logic over usual propositional non-modal language. $\mathsf{IPC}_\Box$ is the same theory IPC in the extended language of propositional modal language, i.e. its language is propositional modal language and its axioms and rules are the same as in IPC. Because we have no axioms for $\Box$ in $\mathsf{IPC}_\Box$, it is obvious that $\Box A$ for each A, behave exactly like an atomic variable inside $\mathsf{IPC}_\Box$. Note that nothing more than the symbol of A plays a role in $\Box A$. The first-order intuitionistic theory is denoted with IQC and CQC is its classical closure, i.e. IQC plus the principle of excluded middle. We have the usual first-order language of arithmetic which has a primitive recursive function symbol for each primitive recursive function. We use the same notations and definitions for Heyting's arithmetic HA as in [Troelstra and van Dalen, 1988], and Peano Arithmetic PA is HA plus the principle of excluded middle. For a set of sentences and rules $\Gamma \cup \{A\}$ in propositional non-modal, propositional modal or first-order language, $\Gamma \vdash A$ means that A is derivable from Γ in the system $\mathsf{IPC}, \mathsf{IPC}_\Box, \mathsf{IQC}$, respectively. For an arithmetical formula A, $\ulcorner A \urcorner$ represents the Gödel number of A. For an arbitrary arithmetical theory T with a set of Δ_0-axioms, we have the Δ_0-predicate $\mathsf{Prov}_T(x, \ulcorner A \urcorner)$, that is a formalization of "x is code of a proof for A in T". We also have the provability predicate $\mathsf{Pr}_T(\ulcorner A \urcorner) := \exists x\ \mathsf{Prov}_T(x, \ulcorner A \urcorner)$.

Definition 3.1. *Suppose T is an r.e. arithmetical theory and σ is a function from atomic variables to arithmetical sentences. We extend σ to all modal propositions A, inductively:*

- $\sigma_T(A) := \sigma(A)$ *for atomic A,*
- σ_T *distributes over $\wedge, \vee, \to$,*
- $\sigma_T(\Box A) := Pr_T(\ulcorner \sigma_T(A) \urcorner)$*, in which $Pr_T(x)$ is the Σ_1-predicate that formalizes T-provability of a sentence with Gödel number x.*

We call σ to be a Σ_1-substitution, if for every atomic p, $\sigma(p)$ be a Σ_1-formula.

Definition 3.2. *Provability logic of a sufficiently strong theory T is defined as*

$$\mathsf{PL}(T) := \{A \in \mathcal{L}_\Box : \forall \sigma\, T \vdash \sigma_T(A)\}.$$

Also the Σ_1-provability logic of T is defined as

$$\mathsf{PL}(T) := \{A \in \mathcal{L}_\Box : \textit{for all } \Sigma_1\textit{-substitution } \sigma \textit{ we have } T \vdash \sigma_T(A)\}.$$

4 NNIL propositions

The class of *No Nested Implications to the Left*, NNIL formulae in the nonmodal language $\mathcal{L}_0$, was introduced in [Visser et al., 1995], and more explored in [Visser, 2002]. The crucial result of [Visser, 2002] is to provide an algorithm (section 4.1) that takes $A \in \mathcal{L}_0$ and returns its best NNIL approximation A^* from below, i.e., $\mathsf{IPC} \vdash A^* \to A$ and for all NNIL formulae B such that $\mathsf{IPC} \vdash B \to A$, we have $\mathsf{IPC} \vdash B \to A^*$. Also for all Σ_1-substitutions σ, we have $\mathsf{HA} \vdash \sigma_{\mathsf{HA}}(\Box A \leftrightarrow \Box A^*)$ [Visser, 2002]. The class NNIL of propositions in $\mathcal{L}_\Box$ is defined inductively:

- NNIL contains all atomic and boxed propositions,
- if $A, B \in \mathsf{NNIL}$, then $A \vee B, A \wedge B, \Box A \in \mathsf{NNIL}$,
- if all $\to$ occurred in A are contained in the scope of a $\Box$ and $B \in \mathsf{NNIL}$, then $A \to B \in \mathsf{NNIL}$.

Before we continue with the NNIL algorithm, let us present a crucial definition from [Visser, 2002]:

Definition 4.1. For any two modal propositions $A, B \in \mathcal{L}_\Box$, define $[A]B$ by induction on the complexity of B:

- $[A]p = p$, for atomic p, $\top$ and $\bot$.
- $[A](B_1 \circ B_2) = [A](B_1) \circ [A](B_2)$, for $\circ \in \{\vee, \wedge\}$,
- $[A](B_1 \to B_2) = A \to (B_1 \to B_2)$,
- $[A](\Box B) = \Box B$.

For a set Γ of modal propositions we define $[A]\Gamma := \bigvee_{B \in \Gamma} [A]B$.

4.1 The NNIL-algorithm

For a modal proposition A, the proposition A^* is defined inductively as follows [Visser, 2002]:

1. A is atomic or boxed, take $A^* := A$.
2. $A = B \wedge C$, take $A^* := B^* \wedge C^*$.
3. $A = B \vee C$, take $A^* := B^* \vee C^*$.
4. $A = B \to C$, we have several sub-cases. In the following, an occurrence of E in D is called an *outer occurrence*, if E is neither in the scope of an implication nor in the scope of a boxed formula. Also an outer occurrence of a conjunction is leftmost, if it is the leftmost outer conjunction.
   (a) C contains an outer occurrence of a conjunction. To make the algorithm deterministic, let $D \wedge E$ be the leftmost outer conjunction in C. In this case, there is some formula $J(q)$ such that
      - q is a propositional variable not occurring in A.
      - q is outer in J and occurs exactly once.
      - $C = J[q|(D \wedge E)]$.

      Now set $C_1 := J[q|D], C_2 := J[q|E]$ and $A_1 := B \to C_1, A_2 := B \to C_2$ and finally, define $A^* := A_1^* \wedge A_2^*$.
   (b) B contains an outer occurrence of a disjunction. To make the algorithm deterministic, let $D \vee E$ be the leftmost outer disjunction in B. In this case, there is some formula $J(q)$ such that
      - q is a propositional variable not occurring in A.
      - q is outer in J and occurs exactly once.
      - $B = J[q|(D \vee E)]$.

      Now set $B_1 := J[q|D], B_2 := J[q|E]$ and $A_1 := B_1 \to C, A_2 := B_2 \to C$ and finally, define $A^* := A_1^* \wedge A_2^*$.
   (c) $B = \bigwedge X$ and $C = \bigvee Y$ and X, Y are sets of implications, atomics or boxed formulas. We have several sub-cases:

i. X contains an atomic variable or a boxed formula E. To make the algorithm deterministic, let E be the leftmost atomic or boxed in B. We set $D := \bigwedge(X \setminus \{E\})$ and take $A^* := E^* \to (D \to C)^*$.
ii. X contains $\top$. Define $D := \bigwedge(X \setminus \{\top\})$ and take $A^* := (D \to C)^*$.
iii. X contains $\bot$. Take $A^* := \top$.
iv. X contains only implications. For any $D = E \to F \in X$, define

$$B{\downarrow}D := \bigwedge((X \setminus \{D\}) \cup \{F\}).$$

Let $Z := \{E \mid E \to F \in X\} \cup \{C\}$ and define:

$$A^* := \bigwedge\{((B{\downarrow}D) \to C)^* | D \in X\} \wedge \bigvee\{([B]E)^* \mid E \in Z\}$$

4.2 The binary relation $\blacktriangleright$

Visser [2002] axiomatizes the NNIL-algorithm via a binary relation $\blacktriangleright_\sigma$. This binary relation is tightly related to the interpretability logic [de Jongh and Veltman, 1990; Visser, 1990; Berarducci, 1990; Shavrukov, 1988] and preservativity logic [Iemhoff, 2003, 2001]. We use some modal variant of Visser's $\blacktriangleright_\sigma$. The relation $\blacktriangleright_\sigma$ is defined to be the smallest relation $\blacktriangleright$ on modal propositions in $\mathcal{L}_\Box$ satisfying A1-A4, B1-B3: [Ardeshir and Mojtahedi, 2018]

$A1$. If $\mathsf{iK4} \vdash A \to B$, then $A \blacktriangleright B$.

A2. If $A \blacktriangleright B$ and $B \blacktriangleright C$, then $A \blacktriangleright C$.

A3. If $C \blacktriangleright A$ and $C \blacktriangleright B$, then $C \blacktriangleright A \wedge B$.

A4. If $A \blacktriangleright C$ and $B \blacktriangleright C$, then $A \vee B \blacktriangleright C$.

B1. If $A \blacktriangleright A$, then $\Box A \blacktriangleright \Box B$.

B2. Let X be a set of implications, $B := \bigwedge X$ and $A := B \to C$. Also assume that $Z := \{E | E \to F \in X\} \cup \{C\}$. Then $A \blacktriangleright [B]Z$,

B3. If $A \blacktriangleright B$, then $(C \to A) \blacktriangleright (C \to B)$, for atomic or boxed C.

The following theorem, states that the NNIL-algorithm only uses the axioms and rules presented in the above inductive definition of $\blacktriangleright_\sigma$:

Theorem 4.2. *For every $A \in \mathcal{L}_\Box$ we have $A^* \in \mathsf{NNIL}$, $\mathsf{IPC}_\Box \vdash A^* \to A$ and $A \blacktriangleright_\sigma A^*$. Moreover, if $B \in \mathsf{NNIL}$ is such that $\mathsf{IPC}_\Box \vdash B \to A$, then $\mathsf{IPC}_\Box \vdash B \to A^*$.*

Proof. See [Visser, 2002, Section 7] or [Ardeshir and Mojtahedi, 2018, theorem 4.5]. ❏

Corollary 4.3. *For every $A \in \mathsf{NNIL}$, we have $\mathsf{IPC}_\Box \vdash A \leftrightarrow A^*$.*

Proof. By theorem 4.2 we have $\mathsf{IPC}_\Box \vdash A^* \to A$. On the other hand, since $A \in \mathsf{NNIL}$ and $\mathsf{IPC}_\Box \vdash A \to A$, theorem 4.2 implies $\mathsf{IPC}_\Box \vdash A \to A^*$, as desired. ❏

One interesting consequence of theorem 4.2 follows by taking $B = \top$: $\mathsf{IPC}_\Box \vdash A$ implies $\mathsf{IPC}_\Box \vdash A^*$. However we may strengthen this to the following stronger form:

Corollary 4.4. *For every $A, B \in \mathcal{L}_\Box$, if $\mathsf{IPC}_\Box \vdash A \to B$, then $\mathsf{IPC}_\Box \vdash A^* \to B^*$.*

Proof. Let $\mathsf{IPC}_\Box \vdash A \to B$. By theorem 4.2 we have $\mathsf{IPC}_\Box \vdash A^* \to A$ and hence $\mathsf{IPC}_\Box \vdash A^* \to B$. Again by theorem 4.2 we have $\mathsf{IPC}_\Box \vdash A^* \to B^*$. ❑

The following lemma states that the NNIL-algorithm does not change the sign of occurrences of boxed propositions. We will use this fact later in sections 4.3 and 8.

Lemma 4.5. *Let $A, B \in \mathcal{L}_\Box$ and $\Box B$ be a subformula of A^*. Then $\Box B$ is also a subformula of A. Moreover if $\Box B$ occurs negatively (positively) in A^*, then it also occurs negatively (positively) in A.*

Proof. The proof is by induction on the complexity of A. One must consider all cases separately as explained in the above algorithm, and we leave this long and tedious observation to the reader. ❑

4.3 $\mathsf{NNIL}^{\Box}$ propositions

A proposition $A \in \mathcal{L}_\Box$ belongs to $\mathsf{NNIL}^{\overrightarrow{\Box}}$, if for any positive occurrence of a boxed proposition $\Box B$ in A, we have $B \in \mathsf{NNIL}$. We say that $A \in \mathcal{L}_\Box$ belongs to $\mathsf{NNIL}^{\overleftarrow{\Box}}$, if for any negative occurrence of a boxed proposition $\Box B$ in A, we have $B \in \mathsf{NNIL}$. Finally we say that A belongs to $\mathsf{NNIL}^{\Box}$, if it belongs to both $\mathsf{NNIL}^{\overrightarrow{\Box}}$ and $\mathsf{NNIL}^{\overleftarrow{\Box}}$.
Then define $A^{\overrightarrow{\boxast}}$, $A^{\overleftarrow{\boxast}}$ and $A^{\boxast}$ inductively as follows. These translations will nest the translation $(.)^*$ inside positive, negative and all occurrences of boxes, respectively.

- $p^{\overrightarrow{\boxast}} := p^{\overleftarrow{\boxast}} := p^{\boxast} := p$ for atomic p,
- $(\Box B)^{\overrightarrow{\boxast}} := \Box((B^{\overrightarrow{\boxast}})^*)$, and $(\Box B)^{\overleftarrow{\boxast}} := \Box(B^{\overleftarrow{\boxast}})$ and $(\Box B)^{\boxast} := \Box((B^{\boxast})^*)$,
- $(B \circ C)^{\overrightarrow{\boxast}} := B^{\overrightarrow{\boxast}} \circ C^{\overrightarrow{\boxast}}$, $(B \circ C)^{\overleftarrow{\boxast}} := B^{\overleftarrow{\boxast}} \circ C^{\overleftarrow{\boxast}}$ and $(B \circ C)^{\boxast} := B^{\boxast} \circ C^{\boxast}$, for $\circ \in \{\vee, \wedge\}$,
- $(B \to C)^{\overrightarrow{\boxast}} := B^{\overleftarrow{\boxast}} \to C^{\overrightarrow{\boxast}}$, $(B \to C)^{\overleftarrow{\boxast}} := B^{\overrightarrow{\boxast}} \to C^{\overleftarrow{\boxast}}$ and $(B \to C)^{\boxast} := B^{\boxast} \to C^{\boxast}$.

Roughly speaking, one must replace every negatively boxed subformulas of A with its $(.)^*$ translation to calculate $A^{\overleftarrow{\boxast}}$. We have a similar description for $A^{\overrightarrow{\boxast}}$ and $A^{\boxast}$.

Remark 4.6. *By induction on the complexity of A, one may observe that*

$$A^{\boxast} = \left(A^{\overrightarrow{\boxast}}\right)^{\overleftarrow{\boxast}} = \left(A^{\overleftarrow{\boxast}}\right)^{\overrightarrow{\boxast}}$$

Remark 4.7. *If $A \in \mathsf{NNIL}$ then $A^{\overrightarrow{\boxast}} \in \mathsf{NNIL}$ $A^{\overleftarrow{\boxast}} \in \mathsf{NNIL}$ and $A^{\boxast} \in \mathsf{NNIL}$.*

The following two lemmas will be used in theorem 8.10 where we prove the essential properties of the embedding $(.)^{\boxed{h}}$ from iH_σ to $\mathsf{iGLC_a}$.

Lemma 4.8. *For every $A \in \mathcal{L}_\Box$, we have $A^{\overrightarrow{\boxast}} \in \mathsf{NNIL}^{\overrightarrow{\Box}}$ and $A^{\overleftarrow{\boxast}} \in \mathsf{NNIL}^{\overleftarrow{\Box}}$.*

Proof. We prove both statements simultaneously by induction on the complexity of A.

- A is atomic: Trivial.

- $A = \Box B$: Then $A^{\overleftarrow{\boxast}} = \Box B^{\overleftarrow{\boxast}}$. Consider some negative occurrence of $\Box C$ in $A^{\overleftarrow{\boxast}}$. Then $\Box C$ occurs negatively in $B^{\overleftarrow{\boxast}}$ and hence by induction hypothesis, $C \in \mathsf{NNIL}$. This argument shows that $A^{\overleftarrow{\boxast}} \in \mathsf{NNIL}^{\overleftarrow{\Box}}$, as desired. Next we show that $A^{\overrightarrow{\boxast}} = \Box(B^{\overrightarrow{\boxast}})^* \in \mathsf{NNIL}^{\overrightarrow{\Box}}$. Consider some positive occurrence of a boxed formula $\Box C$ in $A^{\overrightarrow{\boxast}}$. If $C = (B^{\overrightarrow{\boxast}})^*$, by theorem 4.2 we have $C \in \mathsf{NNIL}$. Otherwise, $\Box C$ positively occurs in $(B^{\overrightarrow{\boxast}})^*$. Lemma 4.5 implies that $\Box C$ occurs positively in $B^{\overrightarrow{\boxast}}$ and hence by induction hypothesis $C \in \mathsf{NNIL}$.

- $A = B \circ C$ for $\circ \in \{\vee, \wedge\}$. In this case, every positive (negative) occurrence of $\Box E$ in $A^{\overrightarrow{\boxast}}$ $(A^{\overleftarrow{\boxast}})$, either positively (negatively) occurs in $B^{\overrightarrow{\boxast}}$ $(B^{\overleftarrow{\boxast}})$ or in $C^{\overrightarrow{\boxast}}$ $(C^{\overleftarrow{\boxast}})$. Hence by induction hypothesis we have $E\mathsf{NNIL}$, as desired.

- $A = B \to C$. If $\Box E$ occurs positively in $A^{\overrightarrow{\boxast}}$, then either it negatively occurs in $B^{\overleftarrow{\boxast}}$ or positively occurs in $C^{\overrightarrow{\boxast}}$. Hence induction hypothesis implies $E \in \mathsf{NNIL}$, as desired. With a similar argument, one may show that $E \in \mathsf{NNIL}$ when $\Box E$ negatively occurs in $A^{\overleftarrow{\boxast}}$. ❑

For arbitrary $A \in \mathcal{L}_\Box$ we have:

- $A^* \in \mathsf{NNIL}$,(Theorem 4.2)
- $A^{\overrightarrow{\boxast}} \in \mathsf{NNIL}^{\overrightarrow{\Box}}$, (Lemma 4.8)
- $A^{\overleftarrow{\boxast}} \in \mathsf{NNIL}^{\overleftarrow{\Box}}$, (Lemma 4.8)
- $A^{\boxast} \in \mathsf{NNIL}^{\Box}$, (Lemma 4.8 and remark 4.6)

Lemma 4.9. *For every $A \in \mathcal{L}_\Box$ we have $(A^{\boxast})^* = (A^*)^{\boxast}$.*

Proof. The precise proof, calls for a long and tedious checking of the algorithm of $(.)^*$. We only give some informal argument here and leave details to the reader. One may observe that the algorithm for the computation of $(.)^*$, treats boxed formulas as atomic variables. Also the translation and $(.)^{\boxast}$ only replaces boxed formulas with some other boxed formulas, so its replacements does not affect the computation of $(.)^*$. ❑

5 Leivant's translation

It is well-known that HA has disjunction property, i.e. $\mathsf{HA} \vdash A \vee B$ implies $\mathsf{HA} \vdash A$ or $\mathsf{HA} \vdash B$. For a proof of this fact, see [Troelstra and van Dalen, 1988, theorem 3.5.10]. However HA can not prove the formalization of this fact, i.e. $\mathsf{HA} \nvdash \Box(A \vee B) \to (\Box A \vee \Box B)$, in which $\Box$ must be interpreted as "provability in HA" (see Myhill [1973]; Friedman [1975]). Daniel Leivant then shows that a weaker version of the disjunction property is valid in HA. Leivant [1979] shows $\mathsf{HA} \vdash \Box(A \vee B) \to \Box(\boxdot A \vee \boxdot B)$, in which again $\Box$ must be interpreted as "provability in HA". With almost the same proof, one may extend the Leivant's principle $\Box(A \vee B) \to \Box(\boxdot A \vee \boxdot B)$. To be able to state an extension of Leivant's principle (that is adequate to axiomatize Σ_1-provability logic of HA [Ardeshir and Mojtahedi, 2018]) we need a translation $(.)^l$ on modal language which we name it Leivant's translation and recursively defined as follows.

- $A^l := A$ for atomic A, boxed A or $A = \bot$.
- $(A \wedge B)^l := A^l \wedge B^l$.
- $(A \vee B)^l := \boxdot A^l \vee \boxdot B^l$.
- $(A \to B)^l$ is defined by cases: If $A \in \mathsf{NOI}$, i.e. A has no implication outside the scope of $\Box$, define $(A \to B)^l := A \to B^l$, else define $(A \to B)^l := A \to B$.

The following two lemmas will be used later in section 7 to show that two different axiomatizations for the Σ_1-provability logic of HA are equivalent (theorem 7.2).

Lemma 5.1. *For $A \in \mathsf{NNIL}$ we have $\mathsf{IPC}_\Box \vdash A^l \to A$.*

Proof. Use induction on the complexity of A. All cases are straightforward and left to the reader. ❑

Lemma 5.2. $\mathsf{iK4} \vdash (A^*)^l \to (A^l)^*$.

Proof. Use induction on the complexity of A. All cases are trivial except for $\vee$ and $\to$:

- $A = B \vee C$. Then $(A^*)^l = \boxdot (B^*)^l \vee \boxdot (C^*)^l$. Induction hypothesis implies

  $$\mathsf{iK4} \vdash [\boxdot (B^*)^l \vee \boxdot (C^*)^l] \to [\boxdot (B^l)^* \vee \boxdot (C^l)^*].$$

  Since $\mathsf{IPC}_\Box \vdash E^* \to E$, we have

  $$\mathsf{iK4} \vdash [\boxdot (B^l)^* \vee \boxdot (C^l)^*] \to [((B^l)^* \wedge \Box B^l) \vee ((C^l)^* \wedge \Box C^l)].$$

  The right hand side of the above equation is equal to $(A^l)^*$. Hence $\mathsf{iK4} \vdash (A^*)^l \to (A^l)^*$.

- $A = B \to C$ and $B \in \mathsf{NOI}$. Then it is not difficult to observe that $\mathsf{IPC}_\Box \vdash (A^*)^l \leftrightarrow (B \to (C^*)^l)$. Then by induction hypothesis, $\mathsf{iK4} \vdash (B \to (C^*)^l) \to (B \to (C^l)^*)$. Also the right hand side is $\mathsf{IPC}_\Box$-equivalent to $(A^l)^*$.

- $A = B \to C$ and $B \notin \mathsf{NOI}$. By definition, $A^l = A$. Also lemma 5.1 implies $\mathsf{IPC}_\Box \vdash (A^*)^l \to A^*$. Hence $\mathsf{IPC}_\Box \vdash (A^*)^l \to (A^*)^l$, as desired. ❑

6 Gödel's translation and Heyting's Normal Form

The following translation, is some variant of the Gödel's celebrated translation for embedding IPC in $\mathsf{S4}$ [Gödel, 1933]. The Gödel's translation is considered as first provability interpretation for the modal operator $\Box$.

Definition 6.1. *For every proposition A in modal propositional language, we define A^g inductively as follows:*

- *$A^g := A$, for atomic or boxed A.*
- *$(A \circ B)^g := A^g \circ B^g$. for $\circ \in \{\vee, \wedge\}$.*
- *$(A \to B)^g := \boxdot (A^g \to B^g)$.*

Then define the nested Gödel's translations $(.)^{\overleftarrow{\boxed{g}}}$, $(.)^{\overrightarrow{\boxed{g}}}$ and $(.)^{\boxed{g}}$ as follows.

- $p^{\overrightarrow{\boxed{g}}} := p^{\overleftarrow{\boxed{g}}} := p^{\boxed{g}} := p$ *for atomic* p,
- $(\Box B)^{\overrightarrow{\boxed{g}}} := \Box((B^{\overrightarrow{\boxed{g}}})^g)$, $(\Box B)^{\overleftarrow{\boxed{g}}} := \Box(B^{\overleftarrow{\boxed{g}}})$ *and* $(\Box B)^{\boxed{g}} := \Box((B^{\boxed{g}})^g)$,
- $(B \circ C)^{\overrightarrow{\boxed{g}}} := B^{\overrightarrow{\boxed{g}}} \circ C^{\overrightarrow{\boxed{g}}}$, $(B \circ C)^{\overleftarrow{\boxed{g}}} := B^{\overleftarrow{\boxed{g}}} \circ C^{\overleftarrow{\boxed{g}}}$ *and* $(B \circ C)^{\boxed{g}} := B^{\boxed{g}} \circ C^{\boxed{g}}$, *for* $\circ \in \{\vee, \wedge\}$,
- $(B \to C)^{\overrightarrow{\boxed{g}}} := B^{\overleftarrow{\boxed{g}}} \to C^{\overrightarrow{\boxed{g}}}$, $(B \to C)^{\overleftarrow{\boxed{g}}} := B^{\overrightarrow{\boxed{g}}} \to C^{\overleftarrow{\boxed{g}}}$ *and* $(B \to C)^{\boxed{g}} := B^{\boxed{g}} \to C^{\boxed{g}}$.

Also define $A^h := (A^*)^g$ *and* $A^{\boxed{h}} := (A^{\boxed{*}})^{\boxed{g}}$. *We say that* A *is in Heyting's normal form* ($A \in \mathsf{HNF}_\sigma$), *if either* A *is not an implication and* $A = B^g$ *for some* $B \in \mathsf{NNIL}$ *or* $A = B^g \to C^g$ *for some* $(B \to C) \in \mathsf{NNIL}$. *Also define* $A \in \mathsf{HNF}_\sigma^{\overrightarrow{\Box}}$ ($A \in \mathsf{HNF}_\sigma^{\overleftarrow{\Box}}$), *if for every positive (negative) occurrence of* $\Box B$ *in* A *we have* $B \in \mathsf{HNF}_\sigma$. *Finally let* $\mathsf{HNF}_\sigma^{\Box} := \mathsf{HNF}_\sigma^{\overrightarrow{\Box}} \cap \mathsf{HNF}_\sigma^{\overleftarrow{\Box}}$.

Remark 6.2. *If* $A \in \mathsf{NNIL}$ *then* $A^g \in \mathsf{NNIL}$. *Also* $B \to C \in \mathsf{NNIL}$ *implies* $B^g \to C^g \in \mathsf{NNIL}$.

Remark 6.3. *By induction on the complexity of* A, *one may observe that*

$$A^{\boxed{g}} = \left(A^{\overleftarrow{\boxed{g}}}\right)^{\overrightarrow{\boxed{g}}} = \left(A^{\overrightarrow{\boxed{g}}}\right)^{\overleftarrow{\boxed{g}}}$$

Remark 6.4. *If* $A \in \mathsf{NNIL}$ *then* $A^{\overleftarrow{\boxed{g}}}, A^{\overrightarrow{\boxed{g}}}, A^{\boxed{g}} \in \mathsf{NNIL}$.

The following lemma expresses the main characteristic of a proposition in the range of $(.)^g$.

Lemma 6.5. $\mathsf{iK4C_a} \vdash A^g \to \Box A^g$.

Proof. Use induction on the complexity of A. ❑

The rest of this subsection is devoted to some properties of the Gödel's translation and its nested forms. All these properties will be used later in the paper. The reader may simply skip them and read them whenever it is referenced.
The original definition of the Gödel's translation, puts a $\boxdot$ behind every subformula of A. However we considered a variant of Gödel's translation, for technical reasons. The following property does not hold for the original definition of the translation.

Lemma 6.6. *If* $A \in \mathsf{NOI}$, *then* $A^g = A$.

Proof. Use induction on the complexity of A. ❑

One may easily observe from the definition of $(.)^g$ that every positive occurrence of $\Box B$ in A^g, either positively occurs in A or $B = B_0^g \to B_1^g$ and $B_0 \to B_1$ positively occurs in A. The following lemma is this fact when additionally we have $A \in \mathsf{NNIL}$.

Lemma 6.7. *Let* $A \in \mathsf{NNIL}$ *and* $\Box B$ *positively occurs in* A^g. *Then either* $\Box B$ *positively occurs in* A, *or* $B = B_0^g \to B_1^g$ *for some* $(B_0 \to B_1) \in \mathsf{NNIL}$.

Proof. Use induction on the complexity of A.

- A is atomic, boxed, conjunction or disjunction. These cases are easy and left to the reader.

- $A = E \to F$. Then $A^g = \boxdot(E^g \to F^g)$. Consider a positive occurrence of $\Box B$ in $\boxdot(E^g \to F^g)$. One of the following cases hold:
  - $\Box B = \Box(E^g \to F^g)$. Then $B = E^g \to F^g$ and moreover, since $A \in \mathsf{NNIL}$, we have $E \to F \in \mathsf{NNIL}$.
  - $\Box B$ negatively occurs in E^g. Since $A \in \mathsf{NNIL}$, we have $E \in \mathsf{NOI}$ and hence by lemma 6.6 we have $E^g = E$. Hence $\Box B$ negatively occurs in E, which implies that $\Box B$ positively occurs in A, as desired.
  - $\Box B$ positively occurs in F^g. Then induction hypothesis for F, implies the desired result. ❑

Lemma 6.8. *For every $A \in \mathsf{NNIL}^{\Box}$, we have $A^{\overrightarrow{\boxed{g}}} \in \mathsf{HNF}_\sigma^{\overrightarrow{\Box}}$ and $A^{\overleftarrow{\boxed{g}}} \in \mathsf{HNF}_\sigma^{\overleftarrow{\Box}}$.*

Proof. We prove both statements by induction on the complexity of A.

- A is atomic: trivial.
- $A = \Box B$: Then $A^{\overleftarrow{\boxed{g}}} = \Box(B^{\overleftarrow{\boxed{g}}})$. Consider some negative occurrence of $\Box E$ in $A^{\overleftarrow{\boxed{g}}}$. Then $\Box E$ also negatively occurs in $B^{\overleftarrow{\boxed{g}}}$. Hence induction hypothesis implies the desired result. For the proof of $A^{\overrightarrow{\boxed{g}}} \in \mathsf{HNF}_\sigma^{\overrightarrow{\Box}}$, consider some positive occurrence of $\Box E$ in $A^{\overrightarrow{\boxed{g}}} = \Box((B^{\overrightarrow{\boxed{g}}})^g)$. If $E = (B^{\overrightarrow{\boxed{g}}})^g$, since $B \in \mathsf{NNIL}$ and hence $B^{\overrightarrow{\boxed{g}}} \in \mathsf{NNIL}$, we are done. Otherwise, $\Box E$ positively occurs in $(B^{\overrightarrow{\boxed{g}}})^g$. Lemma 6.7 implies either $\Box E$ positively occurs in $B^{\overrightarrow{\boxed{g}}}$, or $E = E_0^g \to E_1^g$ for some $(E_0 \to E_1) \in \mathsf{NNIL}$. In the first case by induction hypothesis we have the desired result. In the second case, by definition we have $E \in \mathsf{HNF}_\sigma$.
- $A = B \circ C$ for $\circ \in \{\vee, \wedge, \to\}$. Easy by induction hypothesis and left to the reader. ❑

Lemma 6.9. *For every $A \in \mathsf{NNIL}$, we have $\mathsf{iK4} \vdash A^g \to A$.*

Proof. The proof can be carried out by induction on the complexity of A. We only treat the case for implication. Let $A = B \to C$. Since $A \in \mathsf{NNIL}$, we have $B \in \mathsf{NOI}$. Lemma 6.6 implies that $A^g = \boxdot(B \to C^g)$. Hence by induction hypothesis for C, we have $\mathsf{iK4} \vdash A^g \to (B \to C)$. ❑

Lemma 6.10. *For $A \in \mathcal{L}_\Box$ we have*

- *If $A \in \mathsf{HNF}_\sigma$, then $A \in \mathsf{NNIL}$.*
- *If $A \in \mathsf{HNF}_\sigma^{\overrightarrow{\Box}}$, then $A \in \mathsf{NNIL}^{\overrightarrow{\Box}}$.*
- *If $A \in \mathsf{HNF}_\sigma^{\overleftarrow{\Box}}$, then $A \in \mathsf{NNIL}^{\overleftarrow{\Box}}$.*
- *If $A \in \mathsf{HNF}_\sigma^{\Box}$, then $A \in \mathsf{NNIL}^{\Box}$.*

Proof. If $A \in \mathsf{HNF}_\sigma$, then by definition, $A = B^g$ for some $B \in \mathsf{NNIL}$. Hence by lemma 6.9 we have $A \in \mathsf{NNIL}$. The other statements, can be proved by induction on the complexity of A, and the first item. ❑

The following theorem, is the key to state that two different axiomatizations iH_σ and iH'_σ for the Σ_1-provability logic of HA are equivalent (see theorem 7.2).

Theorem 6.11. *For every $A \in \mathsf{NNIL}$, we have $\mathsf{iK4C_a} \vdash A^g \leftrightarrow \boxdot A^l$.*

Proof. Use induction on the complexity of A:

- A is atomic. We have $A^g = A^l = A$ and since $\mathsf{iK4C_a} \vdash \boxdot A \leftrightarrow A$, we are done.
- $A = \Box B$. Then $A^g = A^l = A$ and since $\mathsf{iK4} \vdash \boxdot \Box B \leftrightarrow \Box B$, we are done.
- $A = B \wedge C$. Use induction hypothesis and $\mathsf{iK4} \vdash \boxdot (E \wedge F) \leftrightarrow (\boxdot E \wedge \boxdot F)$.
- $A = B \vee C$. First note that lemma 6.5 implies $\mathsf{iK4C_a} \vdash \boxdot E^g \leftrightarrow E^g$, for every $E \in \mathcal{L}_\Box$. Hence in $\mathsf{iK4C_a}$ we have $A^g \leftrightarrow \boxdot A^g$. Then by definition of $(.)^g$, we have $\mathsf{iK4C_a} \vdash A^g \leftrightarrow \boxdot (B^g \vee C^g)$. By induction hypothesis $\mathsf{iK4C_a} \vdash \boxdot (B^g \vee C^g) \leftrightarrow \boxdot (\boxdot B^l \vee \boxdot C^l)$. Hence by definition of $(.)^l$ we have $\mathsf{iK4C_a} \vdash \boxdot (\boxdot B^l \vee \boxdot C^l) \leftrightarrow \boxdot (A^l)$.
- $A = B \to C$. Then by definition of $(.)^g$ we have $A^g = \boxdot (B^g \to C^g)$. By induction hypothesis $\mathsf{iK4C_a} \vdash \boxdot (B^g \to C^g) \leftrightarrow \boxdot (\boxdot B^l \to \boxdot C^l)$. By a simple reasoning in $\mathsf{iK4}$, we have $\mathsf{iK4} \vdash \boxdot (\boxdot B^l \to \boxdot C^l) \leftrightarrow \boxdot (\boxdot B^l \to C^l)$. Hence again by induction hypothesis, $\mathsf{iK4C_a} \vdash \boxdot (\boxdot B^l \to C^l) \leftrightarrow \boxdot (B^g \to C^l)$. Since $A \in \mathsf{NNIL}$, we have $B \in \mathsf{NOI}$ and lemma 6.6 implies $\boxdot (B^g \to C^l) = \boxdot (B \to C^l)$. Finally by definition of $(.)^l$ we have $\boxdot (B \to C^l) = \boxdot A^l$. ❑

Lemma 6.12. *For $A \in \mathsf{NNIL}$ we have $\mathsf{iK4} \vdash (A^{\boxast})^g \leftrightarrow (A^g)^{\boxast}$.*

Proof. We prove this by induction on the complexity of A. All cases are trivial except for $\to$. Let $A = B \to C \in \mathsf{NNIL}$. By definition we have

$$(A^{\boxast})^g = \boxdot ((B^{\boxast})^g \to (C^{\boxast})^g) \quad \text{and} \quad (A^g)^{\boxast} = ((B^g)^{\boxast} \to (C^g)^{\boxast}) \wedge \Box ((B^g)^{\boxast} \to (C^g)^{\boxast})^*$$

Since $A \in \mathsf{NNIL}$, by remarks 6.2 and 4.7 we have $(A^g)^{\boxast} \in \mathsf{NNIL}$ and hence $(B^g)^{\boxast} \to (C^g)^{\boxast} \in \mathsf{NNIL}$. Hence by corollary 4.3 we have $\mathsf{IPC}_\Box \vdash ((B^g)^{\boxast} \to (C^g)^{\boxast})^* \leftrightarrow ((B^g)^{\boxast} \to (C^g)^{\boxast})$. This implies $\mathsf{iK4} \vdash (A^g)^{\boxast} \leftrightarrow \boxdot ((B^g)^{\boxast} \to (C^g)^{\boxast})$. Hence the induction hypothesis implies the desired result. ❑

Lemma 6.13. *For every $A \in \mathcal{L}_\Box$, we have $\mathsf{iK4} \vdash A^g \leftrightarrow (A^g)^g$.*

Proof. Use induction on the complexity of A. The only nontrivial case is $A = B \to C$. Then $A^g = \boxdot (B^g \to C^g)$ and $(A^g)^g = \boxdot ((B^g)^g \to (C^g)^g) \wedge \Box (B^g \to C^g)$. By induction hypothesis, $\mathsf{iK4} \vdash (A^g)^g \leftrightarrow \boxdot (B^g \to C^g) \wedge \Box (B^g \to C^g)$. Hence $\mathsf{iK4} \vdash (A^g)^g \leftrightarrow A^g$, as desired. ❑

7 Axiom schemas and theories

In this section, we gather all definitions for the axiom-schemata and theories and logics which we will need later on. Firs axiom-schemata:

- $\underline{\mathsf{i}}$: all axiom schemas of IPC in the Hilbert style proof system, i.e. the following axiom schemas: (1) $A \to (B \to A)$, (2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$, (3) $A \to (B \to (A \wedge B))$, (4) $(A \wedge B) \to A$, (5) $(A \wedge B) \to B$, (6) $(A \to C) \to ((B \to C) \to ((A \vee B) \to C))$, (7) $A \to (A \vee B)$, (8) $B \to (A \vee B)$.

- $\underline{\mathsf{K}}$: $\Box(A \to B) \to (\Box A \to \Box B)$,
- $\underline{\mathsf{4}}$: $\Box A \to \Box\Box A$,
- The Löb's axiom, $\underline{\mathsf{L\ddot{o}b}}$ or $\underline{\mathsf{L}}$: $\Box(\Box A \to A) \to \Box A$. [Löb, 1955].
- Restriction of Completeness Principle to atomic variables, $\underline{\mathsf{CP_a}}$ or $\underline{\mathsf{C_a}}$: $p \to \Box p$, for atomic p. [Visser, 1982].
- Leivant's Principle, $\underline{\mathsf{Le}}$: $\Box(B \vee C) \to \Box(\Box B \vee C)$. [Leivant, 1975]
- Extended Σ_1-Leivant's Principle, $\underline{\mathsf{Le}_\sigma}$: $\Box A \to \Box A^l$. [Ardeshir and Mojtahedi, 2018]
- Extended Σ_1-Leivant's Principle, (the weak version) $\underline{\mathsf{Le}'_\sigma}$: $\Box A \to \Box A^g$, for every $A \in \mathsf{NNIL}$.
- The Visser's axiom schema $\underline{\mathsf{V}}$: $\Box A \to \Box A^*$.
- For an axiom schema $\underline{\mathsf{A}}$, the axiom schema $\overline{\mathsf{A}}$ indicates the box of every axiom instance of $\underline{\mathsf{A}}$. Also A indicates $\underline{\mathsf{A}} \wedge \overline{\mathsf{A}}$.

All modal systems which will be defined here, only have one inference rule: modus ponens $\dfrac{B \quad B \to A}{A}$. Also, celebrated modal logics, like $\mathsf{K4}$, which have the necessitation rule of inference, $\dfrac{A}{\Box A}$, by abuse of notation, are considered here with the same name and with the same set of theorems, however without the necessitation rule. The reason for this alternate definition of systems, is quite technical. Of course one may define them with the necessitation rule, but at the cost of losing the uniformity of definitions. So in the rest of this paper, all modal systems, are considered with the modus ponens rule of inference. Note that in the presence of the axiom schema 4, one may finitely axiomatize logics such as $\mathsf{iK4}$ and extensions, without necessitation rule.

Consider a list $\mathsf{A}_1, \ldots, \mathsf{A}_n$ of axiom schemas and also L is a modal logic. The notation $\mathsf{A}_1\mathsf{A}_2\ldots\mathsf{A}_n$ ($L\mathsf{A}_1\mathsf{A}_2\ldots\mathsf{A}_n$) will be used in this paper for a modal system containing all axiom instances of all axiom schemas A_i (and all axioms of L), and is closed under modus ponens. This general notation makes things uniform and easy to remember for later usage. However, we make the following exceptions:

- $\mathsf{iGL} := \mathsf{iKL}$,
- $\mathsf{GL} := \mathsf{iKLP}$,
- $\mathsf{iH}_\sigma := \mathsf{iGL} + \mathsf{Le}_\sigma + \mathsf{CP_a} + \mathsf{V}$.
- $\mathsf{iH}'_\sigma := \mathsf{iGL} + \mathsf{Le}'_\sigma + \mathsf{CP_a} + \mathsf{V}$.

The following theorem is the main result in [Ardeshir and Mojtahedi, 2018]:

Theorem 7.1. *The Σ_1-Provability logic of* HA *is* iH_σ*, i.e.* $\mathsf{PL}_\Sigma(\mathsf{HA}) = \mathsf{iH}_\sigma$.

Proof. For the proof, see [Ardeshir and Mojtahedi, 2018], Theorems 8.1 and 8.2. ❑

Theorem 7.2. $\mathsf{iH}_\sigma = \mathsf{iH}'_\sigma$.

Proof. Theorem 6.11 implies that iH'_σ is included in iH_σ. To show iH_σ is included in iH'_σ, it is enough to show that $\mathsf{iH}'_\sigma \vdash \Box A \to \Box A^l$ for every $A \in \mathcal{L}_\Box$. By Le'_σ, we have $\mathsf{iH}'_\sigma \vdash \Box A^* \to \Box(A^*)^g$. Then theorem 6.11 implies $\mathsf{iH}'_\sigma \vdash \Box A^* \to \Box(A^*)^l$. Lemma 5.2 implies $\mathsf{iH}'_\sigma \vdash \Box A^* \to \Box(A^l)^*$. Since by theorem 4.2 we have $\mathsf{iH}'_\sigma \vdash \Box E \leftrightarrow \Box E^*$, we may conclude $\mathsf{iH}'_\sigma \vdash \Box A \to \Box A^l$, as desired. ❑

8 Embedding of iH'_σ in $\mathsf{iGLC_a}$

In this section, we show (theorem 8.10) that $(.)^{\boxed{h}}$ embeds iH_σ, the Σ_1-provability logic of HA, in $\mathsf{iGLC_a}$, the intuitionistic counterpart of the Σ_1-provability logic of PA. Moreover we show (theorem 8.13 and corollary 8.18) that iH_σ is $\mathsf{HNF}_\sigma^{\square}$-conservative and also $\mathsf{HNF}_\sigma^{\overleftarrow{\square}}$-conservative extension of $\mathsf{iGLC_a}$. (For definitions of $\mathsf{HNF}_\sigma^{\overleftarrow{\square}}$ and $\mathsf{HNF}_\sigma^{\square}$, see definition 6.1.) The following lemma is a first step toward theorem 8.10, the embedding theorem. It states that $(.)^{\boxed{\ast}}$ preserves proofs in $\mathsf{iK4}$ and some extensions of it.

Lemma 8.1. $\mathsf{iK4X}$ *for any* $\mathsf{X} \subseteq \{\mathsf{CP_a}, \mathsf{L\ddot{o}b}, \mathsf{Le}'_\sigma\}$ *is closed under* $(.)^{\boxed{\ast}}$.

Proof. Use induction on the complexity of the proof $\mathsf{iK4} + \mathsf{X} \vdash \varphi$ and prove $\mathsf{iK4} + \mathsf{X} \vdash \varphi^{\boxed{\ast}}$. Note that in this paper, the necessitation rule of inference, is missing. The only inference rule is the modus ponens. Nevertheless, necessitation is admissible in the presence of $\mathsf{4}$ and modus ponens. Having this in our mind, we have only one inductive step: φ is derived by modus ponens from ψ and $\psi \to \varphi$. Then by induction hypothesis, we have $\psi^{\boxed{\ast}}$ and $\psi^{\boxed{\ast}} \to \varphi^{\boxed{\ast}}$. Hence again by modus ponens, we have $\varphi^{\boxed{\ast}}$, as desired.

$\underline{\mathsf{i}}$: Note that for every axiom instance φ for $\mathsf{IPC}_\square$ in the Hilbert style system, $\varphi^{\boxed{\ast}}$ is also an instance for the same axiom schema.

$\bar{\mathsf{i}}$: By theorem 4.2, for every axiom instance φ of $\mathsf{IPC}_\square$, we have $\mathsf{IPC}_\square \vdash \varphi^*$. Since $\varphi^{\boxed{\ast}}$ is an instance for the same axiom schema which φ belongs, we have $\mathsf{IPC}_\square \vdash (\varphi^{\boxed{\ast}})^*$. Hence $\mathsf{iK4} \vdash (\square\varphi)^{\boxed{\ast}}$, as desired.

$\underline{\mathsf{K}}$: We have $\mathsf{IPC}_\square \vdash [(A^{\boxed{\ast}} \to B^{\boxed{\ast}}) \wedge A^{\boxed{\ast}}] \to B^{\boxed{\ast}}$. Hence by corollary 4.4, we have

$$\mathsf{IPC}_\square \vdash [(A^{\boxed{\ast}} \to B^{\boxed{\ast}})^* \wedge (A^{\boxed{\ast}})^*] \to (B^{\boxed{\ast}})^*.$$

This implies $\mathsf{iK4} \vdash [\square(A \to B) \to (\square A \to \square B)]^{\boxed{\ast}}$, as desired.

$\overline{\mathsf{K}}$: Note that

$$\left((\square(A \to B) \to (\square A \to \square B))^{\boxed{\ast}}\right)^* = (\square(A \to B) \to (\square A \to \square B))^{\boxed{\ast}}$$

Then by previous item and necessitation we have

$$\mathsf{iK4} \vdash \square\left((\square(A \to B) \to (\square A \to \square B))^{\boxed{\ast}}\right)^*.$$

$\mathsf{4}$: Easy and left to the reader.

$\mathsf{CP_a}$: Easy and left to the reader.

$\underline{\mathsf{L\ddot{o}b}}$: Since $(\square E \to F)^* = \square E \to F^*$, we have

$$[\square(\square A \to A) \to \square A]^{\boxed{\ast}} = \square\left(\square\left(A^{\boxed{\ast}}\right)^* \to \left(A^{\boxed{\ast}}\right)^*\right) \to \square\left(A^{\boxed{\ast}}\right)^*$$

The right hand side is an instance of the axiom schema $\underline{\mathsf{L\ddot{o}b}}$.

$\overline{\mathsf{L\ddot{o}b}}$: Easy and left to the reader.

$\underline{\mathsf{Le}'_\sigma}$: Let $A \in \mathsf{NNIL}$. Then by remark 4.7 $A^{\boxed{*}} \in \mathsf{NNIL}$. Hence by $\underline{\mathsf{Le}'_\sigma}$ we have $\mathsf{iK4Le}'_\sigma \vdash \Box A^{\boxed{*}} \to \Box(A^{\boxed{*}})^g$. Lemma 6.12 implies $\mathsf{iK4Le}'_\sigma \vdash \Box A^{\boxed{*}} \to \Box(A^g)^{\boxed{*}}$. Since $A \in \mathsf{NNIL}$, by remarks 6.2 and 4.7 we have $(A^g)^{\boxed{*}}, A^{\boxed{*}} \in \mathsf{NNIL}$. Hence by corollary 4.3 we have $\mathsf{IPC}_\Box \vdash ((A^g)^{\boxed{*}})^* \leftrightarrow ((A^g)^{\boxed{*}})$ and $\mathsf{IPC}_\Box \vdash (A^{\boxed{*}})^* \leftrightarrow A^{\boxed{*}}$ which implies $\mathsf{iK4Le}'_\sigma \vdash \Box(A^{\boxed{*}})^* \to \Box((A^g)^{\boxed{*}})^*$. Since $\Box(A^{\boxed{*}})^* \to \Box((A^g)^{\boxed{*}})^* = (\Box A \to \Box A^g)^{\boxed{*}}$, we have $\mathsf{iK4Le}'_\sigma \vdash (\Box A \to \Box A^g)^{\boxed{*}}$.

$\overline{\mathsf{Le}'_\sigma}$: Let $A \in \mathsf{NNIL}$. By previous item we have $\mathsf{iK4Le}'_\sigma \vdash \Box(\Box A \to \Box A^g)^{\boxed{*}}$. Since $(\Box A \to \Box A^g)^{\boxed{*}} \in \mathsf{NNIL}$, we have $\mathsf{iK4Le}'_\sigma \vdash \Box[(\Box A \to \Box A^g)^{\boxed{*}}]^*$. Since $\Box[(\Box A \to \Box A^g)^{\boxed{*}}]^* = [\Box(\Box A \to \Box A^g)]^{\boxed{*}}$, we have the desired result. ❑

In the second step toward theorem 8.10, we show that the translation $(.)^{\boxed{g}}$ respects the proofs in $\mathsf{iK4C_a}$ and $\mathsf{iGLC_a}$.

Lemma 8.2. $\mathsf{iK4C_a}$ *and* $\mathsf{iGLC_a}$ *are closed under the Gödel's nested translation* $(.)^{\boxed{g}}$.

Proof. Use induction on the complexity of the proof $\mathsf{iK4C_a} \vdash \varphi$. We have only one inductive step: φ is derived by modus ponens from ψ and $\psi \to \varphi$. Then by induction hypothesis, we have $\psi^{\boxed{g}}$ and $\psi^{\boxed{g}} \to \varphi^{\boxed{g}}$. Hence again by modus ponens, we have $\varphi^{\boxed{g}}$, as desired. It remains to show here that $\varphi^{\boxed{g}}$ is provable in $\mathsf{iGLC_a}$ if φ is an axiom schema of $\mathsf{iGLC_a}$.

$\underline{\mathsf{i}}$: If φ is an axiom instance of IPC, observe that $\varphi^{\boxed{g}}$ is an axiom instance of the same schema, and hence provable in $\mathsf{IPC}_\Box$.

$\overline{\mathsf{i}}$: Let $\varphi = \Box A$ for some axiom instance A of IPC. Then also $A^{\boxed{g}}$ is an instance of the same axiom schema. Hence by fact 1, we have $\mathsf{iK4C_a} \vdash (A^{\boxed{g}})^g$ and then $\mathsf{iK4C_a} \vdash \Box(A^{\boxed{g}})^g$. Since by definition, $\varphi^{\boxed{g}} = \Box(A^{\boxed{g}})^g$, we have the desired result.

$\underline{\mathsf{K}}$: Let $\varphi = \Box(A \to B) \to (\Box A \to \Box B)$. We have $\varphi^{\boxed{g}} = \Box((A^{\boxed{g}} \to B^{\boxed{g}})^g) \to (\Box(A^{\boxed{g}})^g \to \Box(B^{\boxed{g}})^g)$. Since in $\mathsf{iK4}$ we have $\Box = \Box \boxdot$, we have

$$\mathsf{iK4} \vdash \varphi^{\boxed{g}} \leftrightarrow \Box((A^{\boxed{g}})^g \to (B^{\boxed{g}})^g) \to (\Box(A^{\boxed{g}})^g \to \Box(B^{\boxed{g}})^g).$$

The right hand side is again an instance of the axiom schema $\underline{\mathsf{K}}$. Hence $\mathsf{iK4} \vdash \varphi^{\boxed{g}}$.

$\overline{\mathsf{K}}$: Observe that $\mathsf{iK4} \vdash (\Box\varphi)^{\boxed{g}} \leftrightarrow \Box(\varphi^{\boxed{g}})$, in which φ is as in the previous item. Then use the previous item and necessitation.

$\underline{\mathsf{L\ddot{o}b}}$: Let $\varphi = \Box(\Box A \to A) \to \Box A$. Observe that $\mathsf{iK4} \vdash \varphi^{\boxed{g}} \leftrightarrow [\Box(\Box(A^{\boxed{g}})^g \to (A^{\boxed{g}})^g) \to \Box(A^{\boxed{g}})^g]$. Since the right hand side is also an instance of the axiom schema $\mathsf{L\ddot{o}b}$, we have $\mathsf{iGL} \vdash \varphi^{\boxed{g}}$, as desired.

$\overline{\mathsf{L\ddot{o}b}}$: Observe that $\mathsf{iK4} \vdash (\Box\varphi)^{\boxed{g}} \leftrightarrow \Box\varphi^{\boxed{g}}$ and use necessitation together with previous item. (φ is as defined in previous item.) ❑

Since $A^{\boxed{h}} = (A^{\boxed{\ast}})^{\boxed{g}}$, with lemmas 8.1 and 8.2, we almost have all required preliminaries to prove theorem 8.10. However we also need some technical lemmas which comes afterwards.

Lemma 8.3. $\mathsf{iK4C_aX}$ *for* $\mathsf{X} \subseteq \{\mathsf{L\ddot{o}b}, \mathsf{Le}'_\sigma, \mathsf{V}\}$ *is closed under the translation* $(.)^g$.

Proof. Use induction on the complexity of the proof $\mathsf{iK4C_a} \vdash \varphi$. We have only one inductive step: φ is derived by modus ponens from ψ and $\psi \to \varphi$. Then by induction hypothesis, we have ψ^g and $\psi^g \to \varphi^g$. Hence again by modus ponens, we have φ^g, as desired. It remains to show here that φ^g is provable in $\mathsf{iK4C_aX}$ for every axiom instance φ from $\mathsf{iK4C_aX}$. All cases are trivial except for when φ is an axiom instance of IPC: (1) $A \to (B \to A)$, (2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$, (3) $A \to (B \to (A \wedge B))$, (4) $(A \wedge B) \to A$, (5) $(A \wedge B) \to B$, (6) $(A \to C) \to ((B \to C) \to ((A \vee B) \to C))$, (7) $A \to (A \vee B)$, (8) $B \to (A \vee B)$. We only treat the first axiom here. We have $\mathsf{IPC}_\Box \vdash A^g \to (B^g \to A^g)$. By necessitation $\mathsf{iK4} \vdash \Box A^g \to \Box(B^g \to A^g)$. Then lemma 6.5 implies $\mathsf{iK4C_a} \vdash A^g \to \boxdot(B^g \to A^g)$. Hence $\mathsf{iK4C_a} \vdash (A \to (B \to A))^g$, as desired. ❑

Lemma 8.4. *For every* $A \in \mathcal{L}_\Box$ *we have* $\mathsf{iK4C_a} \vdash (A^g)^{\boxed{g}} \leftrightarrow (A^{\boxed{g}})^g$.

Proof. Use induction on the complexity of A. The only nontrivial case is $A = B \to C$. Then $\left(A^{\boxed{g}}\right)^g = \boxdot\left(\left(B^{\boxed{g}}\right)^g \to \left(C^{\boxed{g}}\right)^g\right)$ and

$$(A^g)^{\boxed{g}} = \left((B^g)^{\boxed{g}} \to (C^g)^{\boxed{g}}\right) \wedge \Box\boxdot\left(\left((B^g)^{\boxed{g}}\right)^g \to \left((C^g)^{\boxed{g}}\right)^g\right).$$

By induction hypothesis we have

$$\mathsf{iK4C_a} \vdash \left((B^g)^{\boxed{g}} \to (C^g)^{\boxed{g}}\right) \leftrightarrow \left(\left(B^{\boxed{g}}\right)^g \to \left(C^{\boxed{g}}\right)^g\right).$$

Lemma 8.3 implies

$$\mathsf{iK4C_a} \vdash \boxdot\left(\left((B^g)^{\boxed{g}}\right)^g \to \left((C^g)^{\boxed{g}}\right)^g\right) \leftrightarrow \boxdot\left(\left(\left(B^{\boxed{g}}\right)^g\right)^g \to \left(\left(C^{\boxed{g}}\right)^g\right)^g\right).$$

Also by Lemma 6.13 we have

$$\mathsf{iK4} \vdash \boxdot\left(\left(\left(B^{\boxed{g}}\right)^g\right)^g \to \left(\left(C^{\boxed{g}}\right)^g\right)^g\right) \leftrightarrow \boxdot\left(\left(B^{\boxed{g}}\right)^g \to \left(C^{\boxed{g}}\right)^g\right).$$

Hence

$$\mathsf{iK4C_a} \vdash (A^g)^{\boxed{g}} \leftrightarrow \left[\left(\left(B^{\boxed{g}}\right)^g \to \left(C^{\boxed{g}}\right)^g\right) \wedge \Box\boxdot\left(\left(B^{\boxed{g}}\right)^g \to \left(C^{\boxed{g}}\right)^g\right)\right].$$

Since in $\mathsf{iK4}$, $\Box\boxdot$ is equivalent to $\Box$, the right hand side of the above formula, is equivalent to $(A^{\boxed{g}})^g$, as desired. ❑

Corollary 8.5. *For every* $A \in \mathcal{L}_\Box$ *we have* $\mathsf{iK4C_a} \vdash \left((A^g)^{\boxed{g}}\right)^g \leftrightarrow \left(A^{\boxed{g}}\right)^g$ *and* $\mathsf{iK4C_a} \vdash \left((A^g)^{\boxed{g}}\right)^g \leftrightarrow (A^g)^{\boxed{g}}$.

Proof. Lemma 8.4 implies $\mathsf{iK4C_a} \vdash (A^g)^{\boxed{g}} \leftrightarrow (A^{\boxed{g}})^g$ and then by lemma 8.3 we have $\mathsf{iK4C_a} \vdash \left((A^g)^{\boxed{g}}\right)^g \leftrightarrow \left((A^{\boxed{g}})^g\right)^g$. Hence by lemma 6.13 we have $\mathsf{iK4C_a} \vdash \left((A^{\boxed{g}})^g\right)^g \leftrightarrow (A^{\boxed{g}})^g$.

For proof of the other statement, by lemma 6.13 we have $\mathsf{iK4} \vdash A^g \leftrightarrow (A^g)^g$. Hence lemma 8.2 implies $\mathsf{iK4C_a} \vdash (A^g)^{\boxed{g}} \leftrightarrow ((A^g)^g)^{\boxed{g}}$. Then lemma 8.4 implies $\mathsf{iK4C_a} \vdash ((A^g)^g)^{\boxed{g}} \leftrightarrow \left((A^g)^{\boxed{g}}\right)^g$. ❑

Lemma 8.6. *Let $A \in \mathcal{L}_\Box$. Every negative (positive) occurrence of $\Box B$ in $A^{\overrightarrow{\boxed{\ast}}}$ (in $A^{\overleftarrow{\boxed{\ast}}}$) also negatively (positively) occurs in A. Moreover, if $A \in \mathsf{NNIL}$, every negative (positive) occurrence of $\Box B$ in $A^{\overrightarrow{\boxed{g}}}$ (in $A^{\overleftarrow{\boxed{g}}}$) also negatively (positively) occurs in A.*

Proof. The proof can be carried out easily by induction on the complexity of A. All cases are trivial except for $A = \Box B$, for which we have the desired result by lemma 4.5. ❑

Corollary 8.7. *For every $A \in \mathcal{L}_\Box$, if $A \in \mathsf{NNIL}^{\overleftarrow{\Box}}$ ($A \in \mathsf{NNIL}^{\overrightarrow{\Box}}$) then $A^{\overrightarrow{\boxed{\ast}}} \in \mathsf{NNIL}^{\overleftarrow{\Box}}$ ($A^{\overleftarrow{\boxed{\ast}}} \in \mathsf{NNIL}^{\overrightarrow{\Box}}$).*

Proof. Use lemma 8.6. ❑

Corollary 8.8. *For every $A \in \mathcal{L}_\Box$, if $A \in \mathsf{HNF}_\sigma^{\overleftarrow{\Box}}$ ($A \in \mathsf{HNF}_\sigma^{\overrightarrow{\Box}}$) then $A^{\overrightarrow{\boxed{\ast}}}, A^{\overrightarrow{\boxed{g}}} \in \mathsf{HNF}_\sigma^{\overleftarrow{\Box}}$ ($A^{\overleftarrow{\boxed{\ast}}}, A^{\overleftarrow{\boxed{g}}} \in \mathsf{HNF}_\sigma^{\overrightarrow{\Box}}$).*

Proof. Use lemma 8.6. ❑

Lemma 8.9. *For every $A \in \mathcal{L}_\Box$, $\mathsf{iH}_\sigma \vdash A$ implies $\mathsf{iGLC_a} \vdash A^{\boxed{h}}$.*

Proof. Let $\mathsf{iH}_\sigma \vdash A$. Hence by theorem 7.2 $\mathsf{iH}'_\sigma \vdash A$. Then $\mathsf{iGLC_aLe}'_\sigma \vdash \bigwedge X \to A$ for some finite set X of instances of the axiom schema V. Then by lemma 8.1 we have $\mathsf{iGLC_aLe}'_\sigma \vdash \bigwedge X^{\boxed{\ast}} \to A^{\boxed{\ast}}$, in which $X^{\boxed{\ast}} := \{B^{\boxed{\ast}} : B \in X\}$. For every $B^{\boxed{\ast}} \in X^{\boxed{\ast}}$ we show $\mathsf{iK4} \vdash B^{\boxed{\ast}}$. We have the following cases:

- $B = \Box E \to \Box E^*$: then $B^{\boxed{\ast}} = \Box(E^{\boxed{\ast}})^* \to \Box((E^*)^{\boxed{\ast}})^*$. Theorem 4.2 implies $E^* \in \mathsf{NNIL}$ and remark 4.7 implies $(E^*)^{\boxed{\ast}} \in \mathsf{NNIL}$. Then corollary 4.3 implies $\mathsf{IPC}_\Box \vdash ((E^*)^{\boxed{\ast}})^* \leftrightarrow (E^*)^{\boxed{\ast}}$. Hence $\mathsf{iK4} \vdash \Box(E^*)^{\boxed{\ast}} \to \Box((E^*)^{\boxed{\ast}})^*$, and then by lemma 4.9 we have $\mathsf{iK4} \vdash \Box(E^{\boxed{\ast}})^* \to \Box((E^*)^{\boxed{\ast}})^*$.

- $B = \Box(\Box E \to \Box E^*)$: use necessitation and the previous item to show $\mathsf{iK4} \vdash B^{\boxed{\ast}}$.

This argument shows that $\mathsf{iGLC_aLe}'_\sigma \vdash A^{\boxed{\ast}}$. Hence there is some finite set Y of the instances of the axiom schema Le'_σ such that $\mathsf{iGLC_a} \vdash \bigwedge Y \to A^{\boxed{\ast}}$. Lemma 8.2 implies $\mathsf{iGLC_a} \vdash \bigwedge Y^{\boxed{g}} \to (A^{\boxed{\ast}})^{\boxed{g}}$. It is enough to show that $\mathsf{iGLC_a} \vdash \bigwedge Y^{\boxed{g}}$. Then we would have $\mathsf{iGLC_a} \vdash (A^{\boxed{\ast}})^{\boxed{g}}$, as desired. Consider some instance of the axiom schema Le'_σ of the form $B = \Box E \to \Box E^g$, in which $E \in \mathsf{NNIL}$. By definition we have $B^{\boxed{g}} = \Box(E^{\boxed{g}})^g \to \Box((E^g)^{\boxed{g}})^g$. Corollary 8.5 implies $\mathsf{iK4C_a} \vdash B^{\boxed{g}} \leftrightarrow [\Box(E^{\boxed{g}})^g \to (E^{\boxed{g}})^g]$ and hence $\mathsf{iK4C_a} \vdash B^{\boxed{g}}$. ❑

Theorem 8.10. *The translation $(.)^{\boxed{h}}$ has the following properties:*

1. $\mathsf{iH}'_\sigma \vdash A \leftrightarrow A^{\boxed{h}}$.

2. $\mathsf{iH}'_\sigma \vdash A$ *iff* $\mathsf{iGLC_a} \vdash A^{\boxed{h}}$.

3. $(A \to B)^{\boxed{h}} = A^{\boxed{h}} \to B^{\boxed{h}}$.

Proof. Item 3 is trivial. Item 2 is a consequence of lemma 8.9 and first item. So it remains to prove item 1. First by induction on the complexity of A, observe that $\mathsf{iH}'_\sigma \vdash A \leftrightarrow A^{\boxed{*}}$. Also again by induction on the complexity of $A \in \mathsf{NNIL}^{\Box}$ we have $\mathsf{iH}'_\sigma \vdash A \leftrightarrow A^{\boxed{g}}$. Then by lemma 4.8 we have $A^{\boxed{*}} \in \mathsf{NNIL}^{\Box}$ and hence $\mathsf{iH}'_\sigma \vdash A \leftrightarrow (A^{\boxed{*}})^{\boxed{g}}$, as desired. ❑

Next we show in theorem 8.13 that iH_σ is an $\mathsf{HNF}^{\Box}_\sigma$-conservative extension of $\mathsf{iGLC_a}$. The following two technical lemmas will be helpful in the proof of theorem 8.13.

Lemma 8.11. *For every $A \in \mathsf{NNIL}^{\Box}$ we have $\mathsf{iK4} \vdash A^{\boxed{*}} \leftrightarrow A$.*

Proof. Use induction on the complexity of A. We only treat the case $A = \Box B$ here and leave other cases to the reader. Since$A \in \mathsf{NNIL}^{\Box}$ and $\Box B$ positively occurs in A, we have $B \in \mathsf{NNIL}$. Also by definition $A^{\boxed{*}} := \Box((B^{\boxed{*}})^*)$. Remark 4.7 implies $B^{\boxed{*}} \in \mathsf{NNIL}$. Then corollary 4.3 implies $\mathsf{IPC}_\Box \vdash (B^{\boxed{*}})^* \leftrightarrow B^{\boxed{*}}$. Then induction hypothesis implies $\mathsf{iK4} \vdash A^{\boxed{*}} \leftrightarrow \Box B$, as desired. ❑

Lemma 8.12. *For every $A \in \mathsf{HNF}^{\Box}_\sigma$ we have $\mathsf{iK4} \vdash A^{\boxed{g}} \leftrightarrow A$.*

Proof. Use induction on the complexity of A. We only treat the case $A = \Box B$ here and leave other cases to the reader. Since $A \in \mathsf{HNF}^{\Box}_\sigma$, and $\Box B$ positively occurs in A, there is some $B_0 \in \mathsf{NNIL}$ such that $B = B_0^g$. Also by definition we have $A^{\boxed{g}} := \Box((B^{\overrightarrow{\boxed{g}}})^g)$. Hence $A^{\boxed{g}} = \Box(((B_0^g)^{\boxed{g}})^g)$ and by corollary 8.5 we have $\mathsf{iK4C_a} \vdash A^{\boxed{g}} \leftrightarrow \Box((B_0^g)^{\boxed{g}})$. Then induction hypothesis for B implies the desired result. ❑

Theorem 8.13. iH_σ *is* $\mathsf{HNF}^{\Box}_\sigma$*-conservative extension of* $\mathsf{iGLC_a}$. *In other words,* iH_σ *and* $\mathsf{iGLC_a}$ *prove the same* $\mathsf{HNF}^{\Box}_\sigma$*-propositions.*

Proof. Let $\mathsf{iH}_\sigma \vdash A$ and $A \in \mathsf{HNF}^{\Box}_\sigma$. By lemma 8.9 we have $\mathsf{iGLC_a} \vdash (A^{\boxed{*}})^{\boxed{g}}$. Lemma 8.11 implies $\mathsf{iK4} \vdash A^{\boxed{*}} \leftrightarrow A$ and hence by lemma 8.2 we have $\mathsf{iK4C_a} \vdash (A^{\boxed{*}})^{\boxed{g}} \leftrightarrow A^{\boxed{g}}$. This implies $\mathsf{iGLC_a} \vdash A^{\boxed{g}}$ and then by lemma 8.12 we have $\mathsf{iGLC_a} \vdash A$, as desired. ❑

Our next goal is to strengthen this conservativity result in this way: iH_σ is $\mathsf{HNF}^{\overleftarrow{\Box}}_\sigma$-conservative over $\mathsf{iGLC_a}$. This is the statement of corollary 8.18. The following lemmas will be used for this aim.

Lemma 8.14. *For every $A \in \mathcal{L}_\Box$, we have*

1. *$A \in \mathsf{NNIL}^{\overrightarrow{\Box}}$ implies $\mathsf{iK4} \vdash A^{\overrightarrow{\boxed{g}}} \to A$,*

2. *$A \in \mathsf{NNIL}^{\overleftarrow{\Box}}$ implies $\mathsf{iK4} \vdash A \to A^{\overleftarrow{\boxed{g}}}$,*

3. *$A \in \mathsf{NNIL}^{\overrightarrow{\Box}}$ implies $\mathsf{iK4C_aLe}_\sigma \vdash A \to A^{\overrightarrow{\boxed{g}}}$,*

4. *$A \in \mathsf{NNIL}^{\overleftarrow{\Box}}$ implies $\mathsf{iK4C_aLe}_\sigma \vdash A^{\overleftarrow{\boxed{g}}} \to A$.*

Proof. Use induction on the complexity of A:

- A is atomic. Then $A^{\overrightarrow{\boxed{g}}} := A^{\overleftarrow{\boxed{g}}} := A$, and trivially we are done.

- $A = \Box B$. Assume that $A \in \mathsf{NNIL}^{\overleftarrow{\Box}}$. Then $(\Box B)^{\overleftarrow{\boxed{g}}} := \Box(B^{\overleftarrow{\boxed{g}}})$, and by induction hypothesis for B, and the necessitation we have items 2 and 4 for A. Next consider $A \in \mathsf{NNIL}^{\overrightarrow{\Box}}$ which implies $B \in \mathsf{NNIL}$ and $B \in \mathsf{NNIL}^{\overrightarrow{\Box}}$. For the proof of item 1, we reason as follows: By induction hypothesis (item 1) we have $\mathsf{iK4} \vdash B^{\overrightarrow{\boxed{g}}} \to B$. Also by lemma 6.9 we have $\mathsf{iK4} \vdash (B^{\overrightarrow{\boxed{g}}})^g \to B^{\overrightarrow{\boxed{g}}}$, and hence $\mathsf{iK4} \vdash (B^{\overrightarrow{\boxed{g}}})^g \to B$. Then by necessitation we have $\mathsf{iK4} \vdash \Box((B^{\overrightarrow{\boxed{g}}})^g) \to \Box B$. Since by definition $(\Box B)^{\overrightarrow{\boxed{g}}} := \Box((B^{\overrightarrow{\boxed{g}}})^g)$, we have $\mathsf{iK4} \vdash (\Box B)^{\overrightarrow{\boxed{g}}} \to \Box B$, which is the desired result (item 1) for $A = \Box B$. It remains to prove item 3 for $A = \Box B$. By induction hypothesis (item 3) for B, we have $\mathsf{iK4C_aLe_\sigma} \vdash B \to B^{\overrightarrow{\boxed{g}}}$. Hence by necessitation, we have $\mathsf{iK4C_aLe_\sigma} \vdash \Box B \to \Box B^{\overrightarrow{\boxed{g}}}$. By axiom schema $\mathsf{Le_\sigma}$, we have $\mathsf{iK4C_aLe_\sigma} \vdash \Box B^{\overrightarrow{\boxed{g}}} \to \Box((B^{\overrightarrow{\boxed{g}}})^l)$ and hence $\mathsf{iK4C_aLe_\sigma} \vdash \Box B^{\overrightarrow{\boxed{g}}} \to \Box(\boxdot(B^{\overrightarrow{\boxed{g}}})^l)$. Since by remark 6.4 we have $B^{\overrightarrow{\boxed{g}}} \in \mathsf{NNIL}$, theorem 6.11 implies $\mathsf{iK4C_aLe_\sigma} \vdash \Box B^{\overrightarrow{\boxed{g}}} \to \Box((B^{\overrightarrow{\boxed{g}}})^g)$. Hence $\mathsf{iK4C_aLe_\sigma} \vdash \Box B \to (\Box B)^{\overrightarrow{\boxed{g}}}$, as desired.

- $A = B \circ C$, and $\circ \in \{\vee, \wedge\}$. Easy by induction hypothesis and left to the reader.

- $A = B \to C$. We only prove item 3. Treatment of other items are similar and we leave them to the reader. Let $A \in \mathsf{NNIL}^{\overrightarrow{\Box}}$. Hence $C \in \mathsf{NNIL}^{\overrightarrow{\Box}}$ and $B \in \mathsf{NNIL}^{\overleftarrow{\Box}}$. By induction hypothesis (item 3) for C, we have $\mathsf{iK4C_aLe_\sigma} \vdash C \to C^{\overrightarrow{\boxed{g}}}$. By item 4 of the induction hypothesis for B, we have $\mathsf{iK4C_aLe_\sigma} \vdash B^{\overleftarrow{\boxed{g}}} \to B$. Hence, $\mathsf{iK4C_aLe_\sigma} \vdash (B \to C) \to (B^{\overleftarrow{\boxed{g}}} \to C^{\overrightarrow{\boxed{g}}})$, and then $\mathsf{iK4C_aLe_\sigma} \vdash A \to A^{\overrightarrow{\boxed{g}}}$, as desired. ❑

Lemma 8.15. *For every $A \in \mathcal{L}_\Box$, we have*

1. $\mathsf{iK4} \vdash A^{\overrightarrow{\boxed{*}}} \to A$,
2. $\mathsf{iK4} \vdash A \to A^{\overleftarrow{\boxed{*}}}$,
3. $\mathsf{iK4V} \vdash A \to A^{\overrightarrow{\boxed{*}}}$,
4. $\mathsf{iK4V} \vdash A^{\overleftarrow{\boxed{*}}} \to A$.

Proof. The proof can be carried out very similar to the proof of lemma 8.14. One must use the axiom schema $\mathsf{V_\sigma}$ instead of $\mathsf{Le_\sigma}$ and also theorem 4.2 instead of lemma 6.9. ❑

Lemma 8.16. *For every $A \in \mathcal{L}_\Box$,*

- $A \in \mathsf{NNIL}^{\overrightarrow{\Box}}$ *implies* $A^{\overrightarrow{\boxed{g}}}, A^{\overleftarrow{\boxed{g}}} \in \mathsf{NNIL}^{\overrightarrow{\Box}}$,
- $A \in \mathsf{NNIL}^{\overleftarrow{\Box}}$ *implies* $A^{\overrightarrow{\boxed{g}}}, A^{\overleftarrow{\boxed{g}}} \in \mathsf{NNIL}^{\overleftarrow{\Box}}$.

Proof. First note that if $A \in \mathsf{NNIL}^{\overrightarrow{\Box}}$ then by lemma 8.6 we have $A^{\overleftarrow{\boxed{g}}} \in \mathsf{NNIL}^{\overrightarrow{\Box}}$. Also if $A \in \mathsf{NNIL}^{\overleftarrow{\Box}}$ then by lemma 8.6 we have $A^{\overrightarrow{\boxed{g}}} \in \mathsf{NNIL}^{\overleftarrow{\Box}}$. Prove the remaining statements by induction on the complexity of A. We only treat the boxed case and leave other cases to reader. Let $A = \Box B$ and $A \in \mathsf{NNIL}^{\overrightarrow{\Box}}$. We show $A^{\overrightarrow{\boxed{g}}} \in \mathsf{NNIL}^{\overrightarrow{\Box}}$. By definition we have $A^{\overrightarrow{\boxed{g}}} := \Box((B^{\overrightarrow{\boxed{g}}})^g)$. Consider some positive occurrence of $\Box C$ in $\Box((B^{\overrightarrow{\boxed{g}}})^g)$. We have two cases:

- $C = (B^{\overrightarrow{\boxed{g}}})^g$. Since $\Box B \in \mathsf{NNIL}^{\overrightarrow{\Box}}$ and $\Box B$ positively occurs in A, we have $B \in \mathsf{NNIL}$. Remark 6.4 implies $B^{\overrightarrow{\boxed{g}}} \in \mathsf{NNIL}^{\overrightarrow{\Box}}$. Then remark 6.2 implies $C \in \mathsf{NNIL}$.

- $\Box C$ positively occurs in $(B^{\overrightarrow{\boxed{g}}})^g$. Since $B^{\overrightarrow{\boxed{g}}} \in \mathsf{NNIL}$, lemma 6.7 we have two sub-cases:

  - $\Box C$ positively occurs in $B^{\overrightarrow{\boxed{g}}}$. By induction hypothesis for B we have $C \in \mathsf{NNIL}$.
  - $C = C_0^g \to C_1^g$ and $C_0 \to C_1 \in \mathsf{NNIL}$. Then by remark 6.2 we have $C \in \mathsf{NNIL}$. ❑

Lemma 8.17. *Let $A \in \mathsf{HNF}_\sigma^{\overleftarrow{\Box}}$. There is some $A' \in \mathsf{HNF}_\sigma^{\Box}$ such that*

- $\mathsf{iK4C_a} \vdash A' \to A$.
- $\mathsf{iH}_\sigma \vdash A \to A'$.

Proof. Let $A' := (A^{\overrightarrow{\boxed{*}}})^{\overrightarrow{\boxed{g}}}$. By lemma 8.15 we have $\mathsf{iK4} \vdash A^{\overrightarrow{\boxed{*}}} \to A$ and $\mathsf{iK4V} \vdash A \to A^{\overrightarrow{\boxed{*}}}$. Since $A^{\overrightarrow{\boxed{*}}} \in \mathsf{NNIL}^{\overrightarrow{\Box}}$, by lemma 8.14 we have $\mathsf{iK4C_a} \vdash (A^{\overrightarrow{\boxed{*}}})^{\overrightarrow{\boxed{g}}} \to A^{\overrightarrow{\boxed{*}}}$ and $\mathsf{iK4C_aLe}_\sigma \vdash A^{\overrightarrow{\boxed{*}}} \to (A^{\overrightarrow{\boxed{*}}})^{\overrightarrow{\boxed{g}}}$. Hence we have $\mathsf{iK4C_a} \vdash A' \to A$. and $\mathsf{iH}_\sigma \vdash A \to A'$. It remains only to show that $A' \in \mathsf{HNF}_\sigma^{\Box}$. Since $A \in \mathsf{HNF}_\sigma^{\overleftarrow{\Box}}$, by lemma 6.10, $A \in \mathsf{NNIL}^{\overleftarrow{\Box}}$. Corollary 8.7 implies $A^{\overrightarrow{\boxed{*}}} \in \mathsf{NNIL}^{\overleftarrow{\Box}}$. By lemma 4.8 we have $A^{\overrightarrow{\boxed{*}}} \in \mathsf{NNIL}^{\overrightarrow{\Box}}$. Lemma 8.16 implies $(A^{\overrightarrow{\boxed{*}}})^{\overrightarrow{\boxed{g}}} \in \mathsf{NNIL}^{\overrightarrow{\Box}}$ and $(A^{\overrightarrow{\boxed{*}}})^{\overrightarrow{\boxed{g}}} \in \mathsf{NNIL}^{\overleftarrow{\Box}}$. Hence $(A^{\overrightarrow{\boxed{*}}})^{\overrightarrow{\boxed{g}}} \in \mathsf{NNIL}^{\Box}$. Lemma 6.8 implies that $(A^{\overrightarrow{\boxed{*}}})^{\overrightarrow{\boxed{g}}} \in \mathsf{HNF}_\sigma^{\overrightarrow{\Box}}$. On the other hand, since $A \in \mathsf{HNF}_\sigma^{\overleftarrow{\Box}}$, corollary 8.8 implies $(A^{\overrightarrow{\boxed{*}}})^{\overrightarrow{\boxed{g}}} \in \mathsf{HNF}_\sigma^{\overleftarrow{\Box}}$. Hence by definition we have $(A^{\overrightarrow{\boxed{*}}})^{\overrightarrow{\boxed{g}}} \in \mathsf{HNF}_\sigma^{\Box}$. ❑

Corollary 8.18. iH_σ *is* $\mathsf{HNF}_\sigma^{\overleftarrow{\Box}}$*-conservative extension of* $\mathsf{iGLC_a}$*, in other words* iH_σ *and* $\mathsf{iGLC_a}$ *prove the same* $\mathsf{HNF}_\sigma^{\overleftarrow{\Box}}$ *propositions.*

Proof. Let $\mathsf{iH}_\sigma \vdash A$ for some $A \in \mathsf{HNF}_\sigma^{\overleftarrow{\Box}}$. Let $A' \in \mathsf{HNF}_\sigma^{\Box}$ as provided by lemma 8.17. Hence $\mathsf{iH}_\sigma \vdash A'$. Theorem 8.13 implies $\mathsf{iGLC_a} \vdash A'$ and hence by lemma 8.17 we have $\mathsf{iGLC_a} \vdash A$. ❑

Future works

Let $\underline{\mathsf{S}}$ indicate the axiom schema $\Box A \to A$ and $\underline{\mathsf{P}}$ be the principle of excluded middle $A \vee \neg A$. [Mojtahedi, 2019] characterizes the truth Σ_1-provability logic of HA as $\mathsf{iH}_\sigma\underline{\mathsf{SP}}$. On the other hand, [Visser, 1982] characterizes the truth Σ_1-provability logic of PA as $\mathsf{GL\underline{S}C_a}$. We conjecture that $\mathsf{iH}_\sigma\underline{\mathsf{SP}}$ is embeddable in the intuitionistic counterpart of $\mathsf{GL\underline{S}C_a}$, i.e. $\mathsf{iGL\underline{SP}C_a}$.

Also [Ardeshir and Mojtahedi, 2019] characterizes the Σ_1-provability logic of HA^*, the self-completion of HA. Also [Visser, 1982] characterizes the Σ_1-provability logic of PA^*, as iGLCT, in which $\underline{\mathsf{T}} := \Box(A \to B) \to (A \vee (A \to B))$. We conjecture that the Σ_1-provability logic of HA^* is embeddable in the intuitionistic counterpart of iGLCT, i.e. iGLC.

However the provability logic of HA is not yet characterized, we conjecture that it is also embeddable in iGL, the intuitionistic counterpart of the provability logic of PA.

References

Ardeshir, M. and Mojtahedi, M. (2014). The de Jongh property for Basic Arithmetic. *Archive for Mathematical Logic*, 53(7-8):881–895.

Ardeshir, M. and Mojtahedi, M. (2018). The Σ_1-Provability Logic of HA. *Annals of Pure and Applied Logic*, 169(10):997–1043.

Ardeshir, M. and Mojtahedi, M. (2019). The Σ_1-Provability Logic of HA^*. *Jornal of Symbolic Logic*, 84(3):118–1135.

Artemov, S. and Beklemishev, L. (2004). Provability logic. In Gabbay, D. and Guenthner, F., editors, *in Handbook of Philosophical Logic*, volume 13, pages 189–360. Springer, 2nd edition.

Beklemishev, L. and Visser, A. (2006). Problems in the logic of provability. In *Mathematical problems from applied logic. I*, volume 4 of *Int. Math. Ser. (N. Y.)*, pages 77–136. Springer, New York.

Berarducci, A. (1990). The Interpretability Logic of Peano Arithmetic. *Journal of Symbolic Logic*, 55(3):1059–1089.

de Jongh, D. (1970). The maximality of the intuitionistic predicate calculus with respect to heyting's arithmetic. *Journal of Symbolic Logic*, 36:606.

de Jongh, D. and Veltman, F. (1990). Provability logics for relative interpretability. In Petkov, P., editor, *Mathematical Logic*, pages 31–41. Springer.

de Jongh, D., Verbrugge, R., and Visser, A. (2011). Intermediate logics and the de jongh property. *Archive for Mathematical Logic*, 50:197–213.

Friedman, H. (1975). The disjunction property implies the numerical existence property. *Proc. Nat. Acad. Sci. U.S.A.*, 72(8):2877–2878.

Gödel, K. (1933). Eine interpretation des intuitionistischen aussagenkalkuls. *Ergebnisse eines mathematischen Kolloquiums*, 4:39–40. English translation in: S. Feferman etal., editors, Kurt Gödel Collected Works, Vol. 1, pages 301-303. Oxford University Press, 1995.

Iemhoff, R. (2001). *Provability Logic and Admissible Rules*. PhD thesis, University of Amsterdam.

Iemhoff, R. (2003). Preservativity Logic. (An analogue of interpretability logic for constructive theories). *Mathematical Logic Quarterly*, 49(3):1–21.

Japardize, G. and de Jongh, D. (1997). The logic of provability. In Buss, S., editor, *Handbook of Proof Theory*, pages 475–545. Elsevier.

Leivant, D. (1975). *Absoluteness in Intuitionistic Logic*. PhD thesis, University of Amsterdam.

Leivant, D. (1979). *Absoluteness of intuitionistic logic*, volume 73 of *Mathematical Centre Tracts*. Mathematisch Centrum, Amsterdam.

Löb, M. (1955). Solution of a Problem of Leon Henkin. *Journal of Symbolic Logic*, 20(2):115–118.

Mojtahedi, M. (2019). Hard Provability Logics. Preprint at https://arxiv.org/abs/1911.04284.

Myhill, J. (1973). A note on indicator-functions. *Proceedings of the AMS*, 39:181–183.

Shavrukov, V. (1988). The logic of relative interpretability over Peano Arithmetic. *Technical report.* Steklov Mathematical Institute, Moscow.

Solovay, R. M. (1976). Provability interpretations of modal logic. *Israel J. Math.*, 25(3-4):287–304.

Troelstra, A. S. and van Dalen, D. (1988). *Constructivism in mathematics. Vol. I*, volume 121 of *Studies in Logic and the Foundations of Mathematics.* North-Holland Publishing Co., Amsterdam.

Visser, A. (1982). On the completeness principle: a study of provability in Heyting's arithmetic and extensions. *Ann. Math. Logic*, 22(3):263–295.

Visser, A. (1990). Interpretability logic. In *Mathematical logic*, pages 175–209. Plenum, New York.

Visser, A. (2002). Substitutions of Σ^0_1 sentences: explorations between intuitionistic propositional logic and intuitionistic arithmetic. *Ann. Pure Appl. Logic*, 114(1-3):227–271. Commemorative Symposium Dedicated to Anne S. Troelstra (Noordwijkerhout, 1999).

Visser, A., van Benthem, J., de Jongh, D., and R. de Lavalette, G. R. (1995). NNIL, a study in intuitionistic propositional logic. In *Modal logic and process algebra (Amsterdam, 1994)*, volume 53 of *CSLI Lecture Notes*, pages 289–326. CSLI Publ., Stanford, CA.

Visser, A. and Zoethout, J. (2019). Provability logic and the completeness principle. *Annals of Pure and Applied Logic*, 170(6):718–753.