

# On Provability Logic of HA

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May 11, 2022

## Abstract

We axiomatize the provability logic of HA and show that it is decidable. Moreover we axiomatize the preservativity and relative admissibility for several modal logics extending iK4. As a main tool, we also provide some sort of semantics, called provability semantics, for modal logics extending iGL, which is a mixture of usual Kripke semantics and provability in propositional modal logics.

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## 1 Introduction

Provability logic is a propositional modal logic with the following intended meaning for  $\Box A$ : “ $A$  is provable in some formal theory  $T$ ”. The first such explicit interpretation for modal operator is [Gödel, 1933], where it uses provability interpretation for the modal operator  $\Box$  in classical modal logic S4 for embedding of intuitionistic propositional logic IPC in S4. This result together with Gödel’s arithmetization of syntax/proof for the aim of incompleteness theorems [Gödel, 1931], are start points to the field of provability logic. Since then in this field of study, many interesting researches have been done and still many problems remains open. We refer the reader to [Beklemishev and Visser, 2006; Artemov and Beklemishev, 2004] for surveys on them. A celebrated result in provability logic, is the characterization of provability logic of Peano Arithmetic PA [Solovay, 1976; Löb, 1955]. More precisely, [Solovay, 1976; Löb, 1955] prove that  $GL \vdash A$  iff for every arithmetical PA-interpretation  $\alpha_{PA}$  we have  $PA \vdash \alpha_{PA}(A)$ , in which GL is the Gödel-Löb logic defined as K4 plus the Löb’s principle  $\Box(\Box A \rightarrow A) \rightarrow \Box A$ . Also  $\alpha_{PA}$  is called PA-interpretation if the following conditions met:

- $\alpha_{PA}(a)$  is an arbitrary first-order sentence in the language of arithmetic for every atomic  $a$ .
- $\alpha_{PA}$  commutes with boolean connectives  $\vee$ ,  $\wedge$  and  $\rightarrow$ .
- $\alpha_{PA}(\Box A)$  is an arithmetization (formalization in the first-order language of arithmetic) for: “ $\alpha_{PA}(A)$  is provable in PA”.

For more details see [Smoryński, 1985; Boolos, 1995]. It is well-known that the mentioned result is robust, i.e. it can be generalized to other strong-enough first-order *classical* theories like  $\mathbf{I}\Delta_0 + \exp$ , ZF and ZFC.

There is another interesting road in provability logic, which considers provability interpretations in Heyting’s Arithmetic, the intuitionistic fragment of PA. First results in this field goes back to [Myhill, 1973; Friedman, 1975], where they prove that  $\Box(B \vee C) \rightarrow (\Box B \vee \Box C)$  does not belong to the provability logic of HA. Nevertheless, [Leivant, 1979] shows that the axiom schema  $\Box(B \vee C) \rightarrow \Box(\Box B \vee \Box C)$  belongs to the provability logic of HA. Then [Visser, 1981, 1982] studies the provability logic of HA and characterizes its letterless fragment. [Visser, 2002; Iemhoff, 2003, 2001c] consider a generalization of provability, called preservativity, which is intuitionistic version of its akin, interpretability [Visser, 1998, 1990]. The propositional language for preservativity, named  $\mathcal{L}_{\triangleright}$  in this paper, has a binary modal operator  $\triangleright$  with following interpretation for  $A \triangleright B$ :

$$\text{For every } \Sigma_1\text{-sentence } S \text{ if } HA \vdash S \rightarrow A \text{ then } HA \vdash S \rightarrow B.$$

Albert Visser axiomatizes a logic, called iPH and together with Dick de Jongh proves the soundness of iPH for above-mentioned arithmetical interpretations. Then Rosalie Iemhoff conjectures that iPH is also complete for the arithmetical interpretations. Although results of this paper reinforce Iemhoff’s conjecture, the arithmetical completeness of iPH remains open. [Ardeshir and Mojtahedi, 2018; Visser and Zoethout, 2019; Ardeshir and Mojtahedi, 2019] characterize the provability logic of HA and  $HA^*$  (a self completion of HA introduced in [Visser, 1982]) for  $\Sigma_1$ -substitutions. In some sense, [Mojtahedi, 2022b] shows that the provability logic of HA for  $\Sigma_1$ -substitutions, named  $iGLC_aH_\sigma$  here,

is essentially  $iGLC_a$ . More precisely, [Mojtahedi, 2022b] defines some translation  $(\cdot)^{\square}$  which embeds  $iGLC_aH_\sigma$  into  $iGLC_a$ . [Mojtahedi, 2021] characterizes  $\Sigma_1$ -provability logics of HA and HA\* relative in PA and the standard model  $\mathbb{N}$ .

In the current paper we axiomatize the provability logic of HA as  $iGL$  plus  $\Box A \rightarrow \Box B$  for every  $A$  and  $B$  with following property (annotated  $A \stackrel{iGL}{\sim}_{C\downarrow SN(\Box)} B$  in this paper):

For every  $E \in C\downarrow SN(\Box)$ , if  $iGL \vdash E \rightarrow A$  then  $iGL \vdash E \rightarrow B$ .

The precise definition of  $C\downarrow SN(\Box)$  is quite technical and might be found in section 2.8. Roughly speaking,  $C\downarrow SN(\Box)$  is the set of all propositions which are projective relative in NNIL and are self-complete.

## Road map

Section 2 includes all elementary and general definitions and their related facts. We also included two main required results in section 2.14. Then in section 3 we elevate the results of [Mojtahedi, 2022a] and axiomatize  $\stackrel{iGL}{\sim}_{C\downarrow SN(\Box)}$  together with several other preservativity/admissibility relationships. In section 4 we provide a Kripke-style semantic, called provability semantic, for which we have soundness and completeness of the provability logic of HA. Finally in section 5 we use provability semantics to reduce arithmetical completeness to its  $\Sigma_1$ -version, in a manner like we had in [Ardeshir and Mojtahedi, 2015].

## 2 Preliminary definitions and facts

This section is mainly devoted to elementary definitions and facts. First we define propositional languages (section 2.1) and substitutions (sections 2.2 and 2.3). Then we define several axiom schemata and propositional logics (section 2.4). Section 2.5 defines two complexity measures  $c_{\rightarrow}(A)$  and  $c_{\rightarrow}^{\square}(A)$ . Kripke semantic for the intuitionistic modal logics is defined in section 2.6. To be self contained, we also proved Kripke completeness of  $iGL$  and  $iGLC_a$  in section 2.6. Section 2.7 defines the Gödel's translation [Gödel, 1933]. Some notations on sets of propositions are defined in section 2.8. Section 2.9 defines the set NNIL of propositions and then states some properties of them. Section 2.10 defines relative admissibility and preservativity and proves some elementary facts about them. Section 2.11 introduces the greatest lower bound relative in some set  $\Gamma$  of propositions and proves some of its elementary properties. In section 2.13 we prove simultaneous fixed-point theorem for  $iGL$ . Finally in section 2.14 we state two salient results from [Ardeshir and Mojtahedi, 2018; Mojtahedi, 2022a] which are crucial for the characterization of the provability logic of HA.

### 2.1 Propositional language

The non-modal propositional language  $\mathcal{L}_0$  includes connectives  $\vee, \wedge, \rightarrow, \perp$  and countably infinite atomic variables  $\text{var} := \{x_1, x_2, \dots\}$  and also countably infinite atomic parameters  $\text{par} := \{p_1, p_2, \dots\}$ . The presence of parameters in the propositional language is quite technical: As we will see later it is better to have some atomics in the propositional language with the intended meaning of  $\Sigma_1$ -sentences. So in the axiomatizations which will be defined in section 2.4 we always have this intended meaning in mind and thus we have the axiom  $p_i \rightarrow \Box p_i$  in  $iK4$ . There will be one more consideration regarding this intended meaning for parameters: We can not substitute parameters arbitrarily. The only permitted substitution for a parameter is such that it does not violate the intended  $\Sigma_1$ -interpretation for it. Negation  $\neg$  is defined as  $\neg A := A \rightarrow \perp$  and  $\top := \perp \rightarrow \perp$ . The union  $\text{var} \cup \text{par}$  is annotated as  $\text{atom}$ , the set of atomics. Also define

$$\mathbb{B} := \{\Box B : B \in \mathcal{L}_0\} \quad \text{and} \quad \text{parb} := \text{par} \cup \mathbb{B} \cup \{\perp\} \quad \text{and} \quad \text{atomb} := \text{parb} \cup \text{var}.$$

We use the notation  $\mathcal{L}_0(X)$  for the set of all boolean combinations of propositions in  $X$ , i.e.  $\mathcal{L}_0(X)$  is the minimum set including  $X \cup \{\perp\}$  which is closed under conjunction, disjunction and implication. The modal language  $\mathcal{L}_\square$  is defined as  $\mathcal{L}_0$  plus the modal unary operator  $\square$ . Also  $\mathcal{L}_\triangleright$  indicates the propositional language  $\mathcal{L}_0$  augmented with a binary modal operator  $\triangleright$ . Whenever we consider the language  $\mathcal{L}_\triangleright$ , we assume that  $\square B := \top \triangleright B$ . Hence in this sense,  $\mathcal{L}_\triangleright$  is an extension of  $\mathcal{L}_\square$ .

## 2.2 Propositional substitutions

A (propositional) substitution  $\theta$  is a function on propositional language which commutes with all connectives. More precisely  $\theta$  satisfies the following conditions:

- $\theta(a)$  is a proposition in the language  $\mathcal{L}_\triangleright$  for every  $a \in \mathbf{atom}$ .
- $\theta(B \circ C) = \theta(B) \circ \theta(C)$  for every  $\circ \in \{\vee, \wedge, \rightarrow\}$ .
- $\theta(\perp) = \perp$ .
- $\theta(B \triangleright C) := \theta(B) \triangleright \theta(C)$ .

By default we assume that all substitutions are identity over  $\mathbf{par}$ . However at some places we need to substitute parameters as well, and of course we will make those places clear to the reader. For a substitution  $\theta$ , the function  $\hat{\theta}$  is defined same as  $\theta$  except for boxed propositions for which  $\hat{\theta}$  operates as identity:

- $\hat{\theta}(B \circ C) = \hat{\theta}(B) \circ \hat{\theta}(C)$  for every  $\circ \in \{\vee, \wedge, \rightarrow\}$ .
- $\hat{\theta}(\perp) = \perp$ .
- $\hat{\theta}(B \triangleright C) := B \triangleright C$ .

We call  $\hat{\theta}$  *outer* substitution. Hence  $\hat{\theta}$  and  $\theta$  are the same in the case of non-modal language.

## 2.3 Arithmetical substitutions

An arithmetical substitution is a function  $\alpha$  on set of atomic variables and parameters  $\mathbf{atom}$  such that  $\alpha(a)$  is a first-order arithmetical sentence for every  $a \in \mathbf{atom}$  and  $\alpha(a) \in \Sigma_1$  for every  $a \in \mathbf{par}$ . Moreover  $\alpha$  is called a  $\Sigma_1$ -substitution if  $\alpha(a) \in \Sigma_1$  for every  $a \in \mathbf{atom}$ .

An arithmetical substitution  $\alpha$  may be extended to  $\mathcal{L}_\triangleright$  as follows:

- $\alpha_{\mathbf{HA}}(a) := \alpha(a)$  for every  $a \in \mathbf{atom}$ , and  $\alpha_{\mathbf{HA}}(\perp) = \perp$ .
- $\alpha_{\mathbf{HA}}$  commutes with boolean connectives:  $\vee, \wedge$  and  $\rightarrow$ .
- $\alpha_{\mathbf{HA}}(A \triangleright B)$  is defined as an arithmetization of following statement:

$$\text{For every } E \in \Sigma_1, \text{ if } \mathbf{HA} \vdash E \rightarrow \alpha_{\mathbf{HA}}(A) \text{ then } \mathbf{HA} \vdash E \rightarrow \alpha_{\mathbf{HA}}(B).$$

Note that above definition for  $\alpha_{\mathbf{HA}}$  is compatible with the well-known provability interpretation for  $\square$  when one assume  $\square B := \top \triangleright B$ .

A strong variant  $\alpha_{\mathbf{HA}}^+$  is defined similarly:

- $\alpha_{\mathbf{HA}}^+(a) := \alpha(a)$  for every  $a \in \mathbf{atom}$ , and  $\alpha_{\mathbf{HA}}^+(\perp) = \perp$ .
- $\alpha_{\mathbf{HA}}^+$  commutes with boolean connectives:  $\vee, \wedge$  and  $\rightarrow$ .
- $\alpha_{\mathbf{HA}}^+(A \triangleright B)$  is defined as  $\varphi$  in conjunction to its provability statement in  $\mathbf{HA}$ , i.e.  $\alpha_{\mathbf{HA}}^+(A \triangleright B) := \varphi \wedge \square_{\mathbf{HA}} \varphi$ , in which  $\varphi$  is an arithmetization of following statement:

$$\text{“for every } E \in \Sigma_1, \text{ if } \mathbf{HA} \vdash E \rightarrow \alpha_{\mathbf{HA}}^+(A) \text{ then } \mathbf{HA} \vdash E \rightarrow \alpha_{\mathbf{HA}}^+(B).”$$

## 2.4 Propositional logics

We consider IPC as the intuitionistic propositional logic over the modal language  $\mathcal{L}_\square$ , i.e. a set of propositions in  $\mathcal{L}_\square$  which is closed under modus ponens ( $\frac{A \quad A \rightarrow B}{B}$ ) and includes all of the following axiom-schemata:

- $A \rightarrow (B \rightarrow A)$ ,
- $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$ ,
- $A \rightarrow (B \rightarrow (A \wedge B))$ ,
- $(A \wedge B) \rightarrow A, (A \wedge B) \rightarrow B$ ,
- $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow ((A \vee B) \rightarrow C))$ ,
- $A \rightarrow (A \vee B), B \rightarrow (A \vee B)$ .

By default, we use  $\vdash$  for derivability in IPC. The following axiom-schemata are defined:

$$\underline{\text{K}}: \square(A \rightarrow B) \rightarrow (\square A \rightarrow \square B).$$

$$\underline{\text{4}}: \square A \rightarrow \square \square A.$$

$$\underline{\text{L}}: \square(\square A \rightarrow A) \rightarrow \square A. \text{ (The Löb's axiom)}$$

$$\underline{\text{C}}_p: p \rightarrow \square p \text{ for every } p \in \text{par}.$$

$$\underline{\text{C}}_a: a \rightarrow \square a \text{ for every } a \in \text{atom}.$$

$\underline{\text{H}}(\Gamma, \text{T})$ :  $\square A \rightarrow \square B$  for every  $A \stackrel{\Gamma}{\vdash} B$ , in which  $\stackrel{\Gamma}{\vdash}$  is defined in section 2.10.

$\underline{\text{H}}$ :  $\underline{\text{H}}(\text{C}\downarrow\text{SN}(\square), \text{iGL})$  in which  $\text{C}\downarrow\text{SN}(\square)$  is as defined in section 2.8.

$\underline{\text{H}}_\sigma$ :  $\underline{\text{H}}(\text{SN}, \text{iGLC}_a)$  in which  $\text{SN}$  is as defined in section 2.8.

$\underline{\text{H}}^\square$ :  $\underline{\text{H}}(\text{SN}(\square), \text{iGL})$  in which  $\text{SN}(\square)$  is as defined in section 2.8.

For an axiom schema  $\underline{\text{X}}$ , let  $\bar{\text{X}}$  indicate  $\square \underline{\text{X}}$  and also  $\text{X}$  indicate  $\underline{\text{X}} \wedge \bar{\text{X}}$ . Given a logic  $\text{L}$  and axiom-schemata  $\text{X}_1, \dots, \text{X}_n$ , the logic  $\text{LX}_1 \dots \text{X}_n$  is defined as  $\text{L}$  plus the axioms  $\text{X}_1, \dots, \text{X}_n$ . Then we define following modal logics:

- $\text{i}$ : IPC plus the box of all axioms in IPC plus  $\text{C}_p$ .
- $\text{iGL} := \text{iK4L}$ . (A sequent calculus for iGL is provided in [van der Giessen and Iemhoff, 2021])

Note that in this setting, iGL and iK4 are closed under necessitation:  $\frac{A}{\square A}$ , and moreover they include the axiom-schema  $\text{C}_p$ . It is not difficult to observe that these extensions of standard iGL and iK4 are equivalent to their standard counterpart, as far as we are working with the standard non-parametric language.

## 2.5 Complexity measures $c_{\rightarrow}(A)$ and $c_{\rightarrow}^{\square}(A)$

Given  $A \in \mathcal{L}_{\square}$  define  $c_{\rightarrow}(A)$  as the maximum number of nested implications which are outside boxes and  $c_{\rightarrow}^{\square}(A)$  is defined as maximum of the number of parameters and boxed subformulas in  $A$  and  $\max\{c_{\rightarrow}(B) : \square B \in \text{sub}(A)\}$ . More precisely, we define  $c_{\rightarrow}(A)$  inductively as follows:

- $c_{\rightarrow}(A) := 0$  for  $A = \square B$  or  $A \in \text{atom}$  or  $A = \perp$ .
- $c_{\rightarrow}(A \circ B) := \max\{c_{\rightarrow}(A), c_{\rightarrow}(B)\}$  for  $\circ \in \{\vee, \wedge\}$ .
- $c_{\rightarrow}(A \rightarrow B) := 1 + \max\{c_{\rightarrow}(A), c_{\rightarrow}(B)\}$ .

Then define

$$c_{\rightarrow}^{\square}(A) := \max(\{c_{\rightarrow}(B) : \square B \in \text{sub}(A)\} \cup \{n_A\}),$$

in which  $n_A$  is defined as the number of elements in  $\text{sub}(A) \cap \text{parb}$ . Remember that  $\text{parb} := \text{par} \cup \{\square B : B \in \mathcal{L}_{\square}\} \cup \{\perp\}$ .

One of the best features of the complexity  $c_{\rightarrow}(A)$  is that there are only finitely many propositions  $A$  with  $c_{\rightarrow}(A) \leq n$ :

**Lemma 2.1.** *Modulo IPC-provable equivalence relation, there are finitely many propositions  $A \in \mathcal{L}_0(X)$  with  $c_{\rightarrow}(A) \leq n$ , in which  $X$  is a finite set of atomic or boxed propositions. Moreover one may effectively compute the finite set of mentioned propositions.*

*Proof.* By induction on  $n$ , we define an upper bound  $f(n)$  for the number of propositions  $A \in \mathcal{L}_0(X)$  with  $c_{\rightarrow}(A) \leq n$ . The computability of such set of propositions is left to the reader.

1.  $f(0)$  : Observe that any  $A$  with  $c_{\rightarrow}(A) = 0$  is IPC-equivalent to a disjunction of conjunctions of propositions in  $X$ . Hence  $f(0) = 2^{2^m}$  is an obvious upper bound, in which  $m$  is the number of propositions in  $X$ .
2.  $f(n+1)$  : For every implication  $B \rightarrow C$  with  $c_{\rightarrow}(B \rightarrow C) \leq n+1$ , we have  $c_{\rightarrow}(B), c_{\rightarrow}(C) \leq n$ , and hence  $f(n)^2$  is an upper bound for the number of inequivalent such propositions. Then since modulo IPC-provable equivalence every proposition is a disjunction of conjunctions of propositions in  $X$  or implications, the following definition is an upperbound:

$$f(n+1) := 2^{[m+f(n)^2]}. \quad \square$$

## 2.6 Kripke models for the intuitionistic modal logics

A Kripke model for the intuitionistic modal logic, is a combination of a Kripke model for intuitionistic logic and the classical modal logic. So as expected, it contains two relations: one ( $\preceq$ ) for the intuitionistic implication and another ( $\square$ ) for the modal operator  $\square$  or  $\triangleright$ . More precisely, a Kripke model is a quadruple  $\mathcal{K} = (W, \preceq, \square, V)$  with the following properties:

- $W \neq \emptyset$ .
- $(W, \preceq)$  is a partial order (transitive and irreflexive). We write  $\preceq$  for the reflexive closure of  $\prec$ .
- $V$  is the valuation on atomics, i.e.  $V \subseteq W \times \text{atom}$ .
- $w \preceq u$  and  $w V a$  implies  $u V a$  for every  $w, u \in W$  and  $a \in \text{atom}$ .
- $(\preceq; \square) \subseteq \square$ , i.e.  $w \preceq u \square v$  implies  $w \square v$ . This condition is assumed to ensure that the previous property holds for all modal propositions and not only for  $a \in \text{atom}$ .

The valuation relation  $V$  could be extended to include all modal propositions as follows:

- $\mathcal{K}, w \Vdash a$  iff  $w V a$ , for  $a \in \text{atom}$ .
- $\mathcal{K}, w \Vdash A \wedge B$  iff  $\mathcal{K}, w \Vdash A$  and  $\mathcal{K}, w \Vdash B$ .
- $\mathcal{K}, w \Vdash A \vee B$  iff  $\mathcal{K}, w \Vdash A$  or  $\mathcal{K}, w \Vdash B$ .
- $\mathcal{K}, w \Vdash A \rightarrow B$  iff for every  $u \succcurlyeq w$  if we have  $\mathcal{K}, u \Vdash A$  then  $\mathcal{K}, u \Vdash B$ .
- $\mathcal{K}, w \Vdash A \triangleright B$  iff for every  $u \sqsupset w$  with  $\mathcal{K}, u \Vdash A$  we have  $\mathcal{K}, w \Vdash B$ .
- $\mathcal{K}, w \Vdash \Box A$  iff for every  $u \sqsupset w$  we have  $\mathcal{K}, u \Vdash A$ .

We also define the following strengthen of  $\Vdash$ :

$$\mathcal{K}, w \Vdash^+ A \quad \text{iff} \quad \text{there is some } u \sqsupset w \text{ such that } \mathcal{K}, u \Vdash A \text{ for every } w' \sqsupset u.$$

We also define the following notions for Kripke models:

- *Finite*: if  $W$  is a finite set.
- *Transitive*: if  $\sqsupset$  is transitive, i.e.  $u \sqsupset v \sqsupset w$  implies  $u \sqsupset w$ .
- *Rooted*: if there is some node  $w_0 \in W$  such that  $w_0 \sqsubseteq w$  for every  $w \in W$ , in which  $u \sqsubseteq v$  iff there is some  $u'$  such that  $u \sqsubseteq u' \preccurlyeq v$ .
- *Conversely well-founded*: if there is no infinite ascending sequence  $w_1 \sqsupset w_2 \sqsupset \dots$ . Note that this condition implies irreflexivity of  $\sqsupset$ .
- *Tree*: if for every  $w \in W$  the set  $\{u \in W : u \preccurlyeq w\}$  is finite linearly ordered (by  $\preccurlyeq$ ) set.
- *Transcendental*: if  $u \sqsupset v$  and  $w \preccurlyeq v$  then  $w = v$ .
- $C_p$ : if  $\mathcal{K} \Vdash C_p$ .
- *Good*: if all of the above properties hold.
- $C_a$ : if  $\mathcal{K} \Vdash C_a$ .

Given two Kripke models  $\mathcal{K} = (W, \preccurlyeq, \sqsupset, V)$  and  $\mathcal{K}' = (W', \preccurlyeq', \sqsupset', V')$ , we say that  $\mathcal{K}'$  is an intuitionistic submodel of  $\mathcal{K}$  (notation  $\mathcal{K}' \leq \mathcal{K}$ ) iff  $W = W'$ ,  $\sqsupset = \sqsupset'$ ,  $V = V'$  and  $\preccurlyeq' \subseteq \preccurlyeq$ . A class  $\mathcal{K}$  of Kripke models has intuitionistic submodel property, if  $\mathcal{K}' \leq \mathcal{K} \in \mathcal{K}$  implies  $\mathcal{K}' \in \mathcal{K}$ . A modal logic  $\Gamma$  is said to have intuitionistic submodel property iff it is sound and complete for some class  $\mathcal{K}$  of good Kripke models with intuitionistic submodel property.

**Theorem 2.2.** *iGL is sound and complete for good Kripke models. Also iGLC<sub>a</sub> is sound and complete for good C<sub>a</sub> Kripke models.*

*Proof.* The soundness parts are straightforward and left to the reader. Also second statement can be easily derived from the first one and left to reader.

Let  $iGL \not\vdash A$  for some  $A \in \mathcal{L}_{\Box}$ . We must find some good Kripke model  $\mathcal{K} = (W, \preccurlyeq, \sqsupset, V)$  such that  $\mathcal{K} \not\vdash A$ . By a standard canonical model construction (see [Iemhoff, 2001a,c]) one may find a finite Kripke model  $\mathcal{K}_0 := (W_0, \preccurlyeq_0, \sqsupset_0, V_0)$  such that  $\mathcal{K}_0, w_0 \not\vdash A$ . Then define

- $W :=$  the set of sequences  $\vec{w} := s_1, w_1, s_2, w_2, s_3, \dots, s_n, w_n$  with the following properties:
  - $w_i \in W_0$  and  $s_i \in \{\prec, \sqsupset\}$  for every  $1 \leq i \leq n$ ,
  - $w_i s_{i+1} w_{i+1}$  for every  $1 \leq i < n$ ,
- Also define the function  $e : W \rightarrow W_0$  as follows. For  $\vec{w} := s_1, w_1, s_2, w_2, s_3, \dots, s_n, w_n$  define  $e(\vec{w}) := w_n$  and also for empty sequence  $\vec{\emptyset}$  define  $e(\vec{\emptyset}) := w_0$ .

- Let  $\vec{w} := s_1, w_1, s_2, w_2, s_3 \dots, s_n, w_n$  and  $\vec{v} := t_1, v_1, t_2, v_2, t_3 \dots, t_m, v_m$ . Then define  $\vec{v} \preceq \vec{w}$  iff the following conditions met:

- $\vec{v}$  is an initial segment of  $\vec{w}$ .
- for every  $m < i \leq n$  we have  $s_i = \prec$ .

Also define  $\vec{v} \sqsubset \vec{w}$  iff the following conditions met:

- $\vec{v}$  is an initial segment of  $\vec{w}$ .
- $s_n = \sqsubset$ .

- $\vec{w} V a$  iff  $e(\vec{w}) V_0 a$ .

It is not difficult to observe that  $\mathcal{K}$  is a Kripke model indeed and moreover it is a good Kripke model. Also by a straightforward induction on the complexity of  $B \in \mathcal{L}_\square$  we may prove that

$$\mathcal{K}, \vec{w} \Vdash B \quad \text{iff} \quad \mathcal{K}_0, e(\vec{w}) \Vdash B. \quad \square$$

**Lemma 2.3.** *Let  $\mathcal{K} = (W, \preceq, \sqsubset, V)$  be a finite irreflexive Kripke model. Then for every  $w_0 \in W$  there is some finite rooted (with the root  $\langle w_0 \rangle$ ) transcendental tree Kripke model  $\mathcal{T} = (W', \preceq', \sqsubset', V')$  which is equivalent to  $\mathcal{K}$ , i.e. there is a function  $e : W' \rightarrow W$  such that  $e(\langle w_0 \rangle) = w_0$  and for every  $w \in W'$  and  $A \in \mathcal{L}_\square$  we have  $\mathcal{T}, w \Vdash A$  iff  $\mathcal{K}, e(w) \Vdash A$ .*

*Proof.* Define  $\mathcal{T} := (W', \preceq', \sqsubset', V')$  as follows.

- $W' :=$  the set of finite sequences (excluding empty sequence)  $\vec{w} := \langle w_0, s_1, w_1, s_2, w_2, s_3 \dots, s_n, w_n \rangle$  with the following properties:

- $w_i \in W$  and  $s_i \in \{\prec, \sqsubset\}$  for every  $1 \leq i \leq n$ .
- $w_i s_{i+1} w_{i+1}$  for every  $0 \leq i < n$ . This means that if  $s_{i+1} = \prec$  then  $w_i \prec w_{i+1}$  and if  $s_{i+1} = \sqsubset$  then  $w_i \sqsubset w_{i+1}$ .

- Also define the function  $e : W' \rightarrow W$  as follows. For  $\vec{w} := \langle w_0, s_1, w_1, s_2, w_2, s_3 \dots, s_n, w_n \rangle$  define  $e(\vec{w}) := w_n$ . Also define  $e(\langle w_0 \rangle) := w_0$ .

- Let  $\vec{v} := \langle v_0, t_1, v_1, t_2, v_2, t_3 \dots, t_m, v_m \rangle$  and  $\vec{w} := \langle w_0, s_1, w_1, s_2, w_2, s_3 \dots, s_n, w_n \rangle$ . Then define  $\vec{v} \preceq' \vec{w}$  iff the following conditions met:

- $\vec{v}$  is an initial segment of  $\vec{w}$ .
- for every  $m < i \leq n$  we have  $s_i = \prec$ .

Also define  $\vec{v} \sqsubset' \vec{w}$  iff the following conditions met:

- $\vec{v}$  is an initial segment of  $\vec{w}$ .
- $n > m$ .
- $s_n = \sqsubset$ .

- $\vec{w} V' a$  iff  $e(\vec{w}) V a$ .

Reasoning that this  $\mathcal{T}$  fulfils all required conditions is left to the reader. □



## 2.7 The Gödel's translation $(.)^\square$

The following translation, is some variant of the Gödel's celebrated translation for the embedding of IPC in S4 [Gödel, 1933].

**Definition 2.4.** For every proposition  $A \in \mathcal{L}_\square$  define  $A^\square$  inductively as follows:

- $A^\square := \Box A$ , for  $A \in \text{var}$ .
- $A^\square := A$  for  $A \in \text{parb}$ .
- $(B \circ C)^\square := B^\square \circ C^\square$ . for  $\circ \in \{\vee, \wedge\}$ .
- $(B \rightarrow C)^\square := \Box(B^\square \rightarrow C^\square)$ .

$A \in \mathcal{L}_\square$  is called self complete if there is some  $B \in \mathcal{L}_\square$  such that  $A = B^\square$ :

$$S := \{B^\square : B \in \mathcal{L}_\square\}.$$

Also  $A$  is called  $\mathsf{T}$ -complete if  $\mathsf{T} \vdash A \rightarrow \Box A$ :

$$C^\mathsf{T} := \{A \in \mathcal{L}_\square : \mathsf{T} \vdash A \rightarrow \Box A\}.$$

Note that for every  $\mathsf{T} \supseteq \text{iK4}$  we have  $S \subseteq C^\mathsf{T}$ . Whenever no confusion is likely, we may omit the superscript  $\mathsf{T}$  in the notation  $C^\mathsf{T}$  and simply write  $C$ .

**Theorem 2.5.**  $\text{iGL}$  and  $\text{iGLC}_a$  are closed under  $(.)^\square$ , i.e. for every  $A \in \mathcal{L}_\square$ ,  $\text{iGL} \vdash A$  ( $\text{iGLC}_a \vdash A$ ) implies  $\text{iGL} \vdash A^\square$  ( $\text{iGLC}_a \vdash A^\square$ ).

*Proof.* Straightforward induction on the proof  $\text{iGL} \vdash A$  ( $\text{iGLC}_a \vdash A$ ) and left to the reader.  $\square$

## 2.8 Notations on set of propositions

In the rest of this paper we deal with several sets of modal propositions. For the simplicity of notations, we write  $X_1 \dots X_n$  for  $X_1 \cap \dots \cap X_n$ , when  $X_i$  are sets of propositions. For example we write  $\text{SN}$  for the set of propositions which are  $\text{N}$  (see section 2.9) and self-complete (as will be defined in this section). Also  $\subseteq_{\text{fin}}$  indicates the finite subset relation. Given  $A \in \mathcal{L}_\square$  let  $\text{sub}(A)$  be the set of all subformulas of  $A$  and  $\text{sub}_o(A)$  is the set of all outer subformulas of  $A$ :

- $\text{sub}_o(a) := \{a\}$  for  $a \in \text{atom}$ .
- $\text{sub}_o(\Box A) := \{\Box A\}$ .
- $\text{sub}_o(B \circ C) := \{B \circ C\} \cup \text{sub}_o(B) \cup \text{sub}_o(C)$ , for  $\circ \in \{\vee, \wedge, \rightarrow\}$ .

Also for arbitrary set  $\Gamma$  of propositions define

- $\text{sub}^\Gamma(A) := \text{sub}(A) \cap \Gamma$ . Hence  $\text{sub}^B(A) := \text{sub}(A) \cap \{\Box B : B \in \mathcal{L}_\square\}$ .
- $\text{sub}_o^\Gamma(A) := \text{sub}_o(A) \cap \Gamma$ . Hence  $\text{sub}_o^B(A) := \text{sub}_o(A) \cap \{\Box B : B \in \mathcal{L}_\square\}$ .
- $\Gamma^\vee := \{\bigvee \Delta : \Delta \subseteq_{\text{fin}} \Gamma \text{ and } \Delta \neq \emptyset\}$ .
- $\downarrow\Gamma :=$  the set of all  $\Gamma$ -projective propositions in the logic  $\mathsf{T}$ . We say that a proposition  $A$  is  $\Gamma$ -projective in  $\mathsf{T}$ , if there is some substitution  $\theta$  and  $B \in \Gamma$  such that  $A \xrightarrow[\mathsf{T}]{\theta} B$ , i.e.  $\mathsf{T} \vdash \hat{\theta}(A) \leftrightarrow B$  and  $\mathsf{T} \vdash A \rightarrow (x \leftrightarrow \theta(x))$  for every  $x \in \text{var}$ . Whenever  $\mathsf{T}$  may be inferred from context, we may omit the superscript  $\mathsf{T}$  and simply write  $\downarrow\Gamma$ .
- $\Gamma(X)$  indicates the set  $\Gamma \cap \mathcal{L}_0(X)$ . Also let  $\Gamma(\Box) := \Gamma(\text{parb})$ .

Also define

- $S := \{B^\square : B \in \mathcal{L}_\square\}$ .
- $C^\top := \{A \in \mathcal{L}_\square : \top \vdash A \rightarrow \square A\}$ . If  $\top$  may be inferred from context, we may omit the  $\top$ -superscript from notation.
- $P^\top := \text{Prime}^\top :=$  the set of all  $\top$ -prime propositions, i.e. the set of propositions  $A$  such that for every  $B, C$  with  $\top \vdash A \rightarrow (B \vee C)$  either we have  $\top \vdash A \rightarrow B$  or  $\top \vdash A \rightarrow C$ . If the logic  $\top$  can be inferred from context, we may omit the superscript  $\top$  from notations.
- Given a set  $\Gamma$  of propositions, define  $\Gamma\text{-NF}_0$  as the set of propositions  $B \in \mathcal{L}_\square$  with either  $B \in \Gamma$  or  $\square B \in \Gamma$ . Then define the set  $\Gamma\text{-NF}$  of propositions in  $\Gamma$ -Normal Form as follows:

$$\Gamma\text{-NF} := \{A \in \mathcal{L}_\square : \forall \square B \in \text{sub}(A) B \in \Gamma\text{-NF}_0\}.$$

And finally we assume that  $(.)^\vee$  has the lowest precedence and  $\downarrow(.)$  has the second lowest precedence. This means that

$$\downarrow \text{SN}(\square)^\vee := (\downarrow(\text{SN}(\square)))^\vee \quad \text{and} \quad C\downarrow \text{SN}(\square)^\vee := (C(\downarrow(\text{SN}(\square))))^\vee.$$

**Lemma 2.6.** *If  $\Gamma$  is a set of  $\top$ -prime propositions then  $\downarrow\Gamma$  is so.*

*Proof.* Let  $E \in \downarrow\Gamma$  such that  $\top \vdash E \rightarrow (B \vee C)$ . Also assume that  $E \xrightarrow[\top]{\theta} E^\dagger \in \Gamma$ . Hence  $\top \vdash E^\dagger \rightarrow (\hat{\theta}(B) \vee \hat{\theta}(C))$  and since  $E^\dagger$  is  $\top$ -prime, either we have  $\top \vdash E^\dagger \rightarrow \hat{\theta}(B)$  or  $\top \vdash E^\dagger \rightarrow \hat{\theta}(C)$ . Hence either we have  $\top, E \vdash \hat{\theta}(E \rightarrow B)$  or  $\top, E \vdash \hat{\theta}(E \rightarrow C)$ . Since  $\theta$  is  $E$ -projective, we have either  $\top, E \vdash E \rightarrow B$  or  $\top, E \vdash E \rightarrow C$ .  $\square$

## 2.9 NNIL propositions

The class of *No Nested Implications to the Left*, NNIL formulae, for the nonmodal language  $\mathcal{L}_0$ , was introduced in [Visser et al., 1995], and more explored in [Visser, 2002]. Here we gathered all we need from [Visser et al., 1995; Visser, 2002]. For simplicity of notations, we may write N for NNIL. The crucial result of [Visser, 2002] is to provide an algorithm that as input, takes  $A \in \mathcal{L}_0$  and returns its best NNIL approximation  $A^*$  from below, i.e.,  $\vdash A^* \rightarrow A$  and for all NNIL formulae  $B$  such that  $\vdash B \rightarrow A$ , we have  $\vdash B \rightarrow A^*$ . Also for all  $\Sigma_1$ -substitutions  $\sigma$ , we have  $\text{HA} \vdash \sigma_{\text{HA}}(\square A \leftrightarrow \square A^*)$  [Visser, 2002]. The classes NNIL and NI of propositions in  $\mathcal{L}_\square$  are defined inductively:

- $A \in \text{NNIL}$  and  $A \in \text{NI}$  for every  $A \in \text{atomb}$ .
- $B \circ C \in \text{NNIL}$  if  $B, C \in \text{NNIL}$ . Also  $B \circ C \in \text{NI}$  if  $B, C \in \text{NI}$ . ( $\circ \in \{\vee, \wedge\}$ )
- $B \rightarrow C \in \text{NNIL}$  if  $B \in \text{NI}$  and  $C \in \text{NNIL}$ .

**Theorem 2.7.** *Let  $A \in \text{NNIL}$  and  $\mathcal{K}' \leq \mathcal{K}$  are two Kripke models. Then  $\mathcal{K} \Vdash A$  implies  $\mathcal{K}' \Vdash A$ .*

*Proof.* Let  $\mathcal{K} = (W, \preceq, \square, V)$  and  $\mathcal{K}' = (W, \preceq', \square, V)$ . First by induction on  $A \in \text{NI}$  show that for every  $w \in W$  we have  $\mathcal{K}, w \Vdash A$  iff  $\mathcal{K}', w \Vdash A$ . Then use induction on  $A \in \text{NNIL}$  and show that for every  $w \in W$ , if  $\mathcal{K}, w \Vdash A$  then  $\mathcal{K}', w \Vdash A$ .  $\square$

**Lemma 2.8.** *Modulo IPC-provable equivalence,  $\text{NNIL}(X)$  is finite, whenever  $X$  is a finite set of atomics or boxed propositions. Moreover one may effectively compute this finite set.*

*Proof.* The decidability of  $\text{NNIL}(X)$  is deduced by decidability of IPC and the following argument for finiteness of  $\text{NNIL}(X)$  and left to the reader. To show that  $\text{NNIL}(X)$  is finite, we will find an upper bound  $f(n)$  for the number of IPC-inequivalent propositions in  $\text{NNIL}(X)$  with  $n$  as the number of elements in  $X$ .

First observe that each proposition in  $\text{NNIL}(X)$  can be written as  $\bigvee \bigwedge C$  in which  $C$  is an atomic, boxed or implication, which we call it a component. Hence  $f(n)$  is less than or equal to  $2^{2^{g(n)}}$ , in which  $g(n)$  is the number of components in  $\text{NNIL}(X)$ . Then observe that  $g(n+1) \leq (n+1)f(n)+n+1$ , because one may assume that each component  $C$  is either of the form  $E \rightarrow A$  for some  $E \in X$  and some  $A$  in  $\text{NNIL}(X \setminus \{E\})$ , or  $C \in X$ . Hence the following (primitive) recursive function is an upperbound for the number of all formulas in  $n$  atomics:

$$f(0) := 2 \quad , \quad f(n+1) := 2^{2^{(n+1)(f(n)+1)}}. \quad \square$$

## 2.10 Admissibility and preservativity

Given a logic  $\mathbb{T}$  we say that an inference rule  $\frac{A}{B}$  is admissible to  $\mathbb{T}$  if  $\mathbb{T} \vdash \theta(A)$  implies  $\mathbb{T} \vdash \theta(B)$  for every substitution  $\theta$ . Characterization of all admissible rules for classical logic is trivial:  $\frac{A}{B}$  is admissible iff  $A \rightarrow B$  is classically valid. However the case for intuitionistic logic IPC or modal logics like K4 are not trivial (see Iemhoff [2001b]; Iemhoff and Metcalfe [2009]; Jeřábek [2005]; Rybakov [1987a,b, 1997]; Goudsmit and Iemhoff [2014]; Iemhoff [2016]). In this paper we deal with a generalization of admissibility. In some sense it is same as admissibility relative in some set  $\Gamma$ . This generalization, has been considered in [Mojtahedi, 2022a] for the propositional language and here in this paper we extend that definition to the modal language. Given a logic  $\mathbb{T}$  and a set  $\Gamma$  of propositions define

$$A \vdash_{\Gamma}^{\mathbb{T}} B \text{ iff for every substitution } \theta \text{ and } C \in \Gamma: \mathbb{T} \vdash \hat{\theta}(C \rightarrow A) \text{ implies } \mathbb{T} \vdash \hat{\theta}(C \rightarrow B).$$

Also we define binary relation between propositions  $\overset{\mathbb{T}}{\approx}$ , the preservativity relation, as follows:

$$A \overset{\mathbb{T}}{\approx} B \text{ iff } \forall E \in \Gamma (\mathbb{T} \vdash E \rightarrow A \Rightarrow \mathbb{T} \vdash E \rightarrow B).$$

[Iemhoff, 2003] studies the preservativity in the first-order language of arithmetic and axiomatizes it via the binary modal operator  $\triangleright$ . Also [Visser, 2002] studies preservativity for propositional non-modal language and among other interesting results, it axiomatizes  $\overset{\text{IPC}}{\approx}_{\text{NNIL}}$ .

**Remark 2.9.** *By definition it can be inferred that  $A \vdash_{\Gamma}^{\mathbb{T}} B$  implies  $A \overset{\mathbb{T}}{\approx} B$ , however the converse may not hold. As a counterexample let  $A$  and  $B$  two different variables and  $\Gamma := \{\top\}$  and  $\mathbb{T} = \text{IPC}$ . Then we have  $A \overset{\text{IPC}}{\approx} B$  and not  $A \vdash_{\Gamma}^{\text{IPC}} B$ .*

**Lemma 2.10.** *Let  $\mathbb{T}$  be a logic which is closed under outer substitutions. Then  $A \vdash_{\Gamma}^{\mathbb{T}} B$  implies  $A \overset{\mathbb{T}}{\approx}_{\Gamma} B$ .*

*Proof.* Let  $A \vdash_{\Gamma}^{\mathbb{T}} B$  and  $E \in \downarrow\Gamma$  with  $\mathbb{T} \vdash E \rightarrow A$ . Assume that  $E \overset{\theta}{\triangleright}_{\mathbb{T}} E^{\dagger} \in \Gamma$ . Since  $\mathbb{T}$  is closed under outer substitutions, we get  $\mathbb{T} \vdash E^{\dagger} \rightarrow \hat{\theta}(A)$ . Then  $A \vdash_{\Gamma}^{\mathbb{T}} B$  implies  $\mathbb{T} \vdash E^{\dagger} \rightarrow \hat{\theta}(B)$ . Hence  $\mathbb{T} \vdash E \rightarrow \hat{\theta}(E \rightarrow B)$  and because  $E \overset{\theta}{\triangleright}_{\mathbb{T}} E^{\dagger}$  we get  $\mathbb{T} \vdash E \rightarrow B$ .  $\square$

Later in this paper we axiomatize  $\overset{\mathbb{T}}{\approx}$  and  $\overset{\mathbb{T}}{\approx}_{\Gamma}$  for several pairs  $(\mathbb{T}, \Gamma)$ . Before we continue with this, let us see some basic axioms and rules.

Given a logic  $\mathbb{T}$ , the logic  $[\mathbb{T}]$  proves statements  $A \triangleright B$  for  $A$  and  $B$  in the language of  $\mathbb{T}$  and has the following axioms and rules:

### Aximos

Ax :  $A \triangleright B$ , for every  $\mathbb{T} \vdash A \rightarrow B$ .

### Rules

$$\frac{A \triangleright B \quad A \triangleright C}{A \triangleright B \wedge C} \text{Conj}$$

$$\frac{A \triangleright B \quad B \triangleright C}{A \triangleright C} \text{Cut}$$

The above mentioned axiom and rules are not interesting, because  $[\mathbb{T}] \vdash A \triangleright B$  iff  $\mathbb{T} \vdash A \rightarrow B$ . However we define several interesting additional rules and axioms as follows. Let  $\Delta \subseteq \mathcal{L}_\square$  and define

Le :  $A \triangleright \square A$  for every  $A \in \mathcal{L}_\square$ .

Le<sup>-</sup> :  $A \triangleright \square A$  for every  $A \in \mathcal{L}_0(\text{parb})$ .

A :  $A \triangleright \hat{\theta}(A)$ , for every substitution  $\theta$ .

V( $\Delta$ ) :  $B \rightarrow C \triangleright \bigvee_{i=1}^{n+m} \{B\}_\Delta(E_i)$ , in which  $B = \bigwedge_{i=1}^n (E_i \rightarrow F_i)$  and  $C = \bigvee_{i=n+1}^{n+m} E_i$ , and

$$\{A\}_\Delta(B) := \begin{cases} B & : B \in \Delta \\ A \rightarrow B & : \text{otherwise} \end{cases}$$

$$\frac{B \triangleright A \quad C \triangleright A}{B \vee C \triangleright A} \text{Disj}$$

$$\frac{A \triangleright B \quad (C \in \Delta)}{C \rightarrow A \triangleright C \rightarrow B} \text{Mont}(\Delta)$$

Finally define

$$[[\mathbb{T}, \Delta]] := [\mathbb{T}] + \text{Disj} + \text{Mont}(\Delta) + \text{V}(\Delta),$$

$$[[\mathbb{T}, \Delta]]\text{Le} := [[\mathbb{T}, \Delta]] + \text{Le} \quad \text{and} \quad [[\mathbb{T}, \Delta]]\text{Le}^- := [[\mathbb{T}, \Delta]] + \text{Le}^-.$$

**Remark 2.11.**  $\mathbb{T} \subseteq \mathbb{T}'$  and  $\Delta \subseteq \Delta'$  implies  $[[\mathbb{T}, \Delta]] \subseteq [[\mathbb{T}', \Delta']]$ .

*Proof.* By induction on the complexity of proof  $[[\mathbb{T}, \Delta]] \vdash A \triangleright A'$  one must show  $[[\mathbb{T}', \Delta']] \vdash A \triangleright A'$ . We only treat here the case V( $\Delta$ ) and leave the rest to reader. So assume that  $A = B \rightarrow C$  and  $A' = \bigvee_{i=1}^{n+m} \{B\}_\Delta(E_i)$ , in which  $B = \bigwedge_{i=1}^n (E_i \rightarrow F_i)$  and  $C = \bigvee_{i=n+1}^{n+m} E_i$ . Since for every  $D, F \in \mathcal{L}_\square$  we have  $\vdash \{D\}_\Delta(F) \rightarrow \{D\}_\Delta(F)$ , we get  $[[\mathbb{T}', \Delta']] \vdash \bigvee_{i=1}^{n+m} \{B\}_\Delta(E_i) \triangleright \bigvee_{i=1}^{n+m} \{B\}_\Delta(E_i)$ . On the other hand by V( $\Delta'$ ) we have  $[[\mathbb{T}', \Delta']] \vdash A \triangleright \bigvee_{i=1}^{n+m} \{B\}_\Delta(E_i)$ . Thus Cut implies desired result.  $\square$

**Theorem 2.12 (Soundness).**  $[[\mathbb{T}]]$  is sound for preservativity interpretations, i.e.  $[[\mathbb{T}]] \vdash A \triangleright B$  implies  $A \Vdash_{\mathbb{T}} B$  for every set  $\Gamma$  of propositions and every logic  $\mathbb{T}$ . Moreover

1. if  $\Gamma$  is  $\mathbb{T}$ -complete, then Le is sound,
2. if  $\Gamma$  is  $\mathbb{T}$ -prime, then Disj is also sound,
3. if  $\Gamma$  is closed under  $\Delta$ -conjunctions, i.e.  $A \in \Gamma$  and  $B \in \Delta$  implies  $A \wedge B \in \Gamma$  (up to  $\mathbb{T}$ -provable equivalence relation), then Mont( $\Delta$ ) is sound.
4. if  $\mathbb{T}$  has intuitionistic submodel property and  $\Gamma \subseteq \text{NNIL}$  and  $\Delta \subseteq \text{atomb}$  then V( $\Delta$ ) is sound.
5. if  $\Gamma \subseteq \mathcal{L}_0(\text{parb})$  and  $\mathbb{T}$  is closed under outer substitutions, then A is also sound.

*Proof.* Easy induction on the complexity of proof  $[[\mathbb{T}]] \vdash A \triangleright B$  and left to the reader. The validity of item 4 is provided by lemma 2.14.  $\square$

**Theorem 2.13 (Soundness).**  $[[\mathbb{T}]]$  is sound for admissibility interpretations, i.e.  $[[\mathbb{T}]] \vdash A \triangleright B$  implies  $A \Vdash_{\mathbb{T}} B$  for every set  $\Gamma$  of propositions and every logic  $\mathbb{T}$  which is closed under outer substitutions. Moreover

1. if  $\Gamma$  is  $\mathbb{T}$ -complete, then  $\text{Le}^-$  is sound,
2. if  $\Gamma$  is  $\mathbb{T}$ -prime, then  $\text{Disj}$  is also sound.
3. if  $\Gamma$  is closed under outer substitutions of  $\Delta$ -conjunctions, i.e.  $A \in \Gamma$  and  $B \in \Delta$  implies  $A \wedge \hat{\theta}(B) \in \Gamma$  (up to  $\mathbb{T}$ -provable equivalence relation), then  $\text{Mont}(\Delta)$  is sound.
4. if  $\mathbb{T}$  has intuitionistic submodel property and  $\Gamma \subseteq \text{NNIL}$  and  $\Delta \subseteq \text{parb}$  then  $\mathbb{V}(\Delta)$  is sound.

*Proof.* Easy induction on the complexity of proof  $[\mathbb{T}] \vdash A \triangleright B$  and left to the reader. The validity of item 4 is provided by lemma 2.14.  $\square$

**Lemma 2.14.** *Let  $\mathbb{T}$  has intuitionistic submodel property and  $\Gamma \subseteq \text{NNIL}$  and  $\Delta \subseteq \text{atomb}$ . Then  $B \rightarrow C \Vdash_{\Gamma}^{\top} \bigvee_{i=1}^{n+m} \{B\}_{\Delta}(E_i)$ , in which  $B = \bigwedge_{i=1}^n (E_i \rightarrow F_i)$  and  $C = \bigvee_{i=n+1}^{n+m} E_i$ . Moreover if  $\Delta \subseteq \text{parb}$  then  $B \rightarrow C \Vdash_{\Gamma}^{\top} \bigvee_{i=1}^{n+m} \{B\}_{\Delta}(E_i)$ .*

*Proof.* We give a contrapositive reasoning only for  $B \rightarrow C \Vdash_{\Gamma}^{\top} \bigvee_{i=1}^{n+m} \{B\}_{\Delta}(E_i)$  and leave the similar argument for  $B \rightarrow C \Vdash_{\Gamma}^{\top} \bigvee_{i=1}^{n+m} \{B\}_{\Delta}(E_i)$  to the reader. Let us fix some class  $\mathcal{M}(\mathbb{T})$  of rooted Kripke models with intuitionistic submodel property for which  $\mathbb{T}$  is sound and complete. Let  $E \in \Gamma$  be such that  $\mathbb{T} \not\vdash E \rightarrow (\bigvee_{i=1}^{n+m} \{B\}_{\Delta}(E_i))$ . Hence there is some  $\mathcal{K} = (W, \preceq, \sqsubset, V) \in \mathcal{M}(\mathbb{T})$  with the root  $w_0$  such that  $\mathcal{K}, w_0 \Vdash E$  and  $\mathcal{K}, w_0 \not\vdash \bigvee_{i=1}^{n+m} \{B\}_{\Delta}(E_i)$ . Let  $I$  be the set of indices  $i$  such that  $E_i \in \Delta$ . Also let  $J$  be the complement of  $I$ . Thus for every  $i \in I$  we have  $\mathcal{K}, w_0 \not\vdash E_i$  and for every  $j \in J$ , there is some  $w_j \succ w_0$  such that  $\mathcal{K}, w_j \Vdash B$  and  $\mathcal{K}, w_j \not\vdash E_j$ . Let  $\preceq'$  defined on  $W$  as follows:

$$\preceq' := \preceq \setminus (\{w_0\} \times \{v \in W : \neg \exists j \in J (w_j \preceq v)\})$$

and define  $\mathcal{K}' := (W, \preceq', \sqsubset, V)$ . Then since  $E \in \text{NNIL}$  and  $\mathcal{K}, w_0 \Vdash E$ , theorem 2.7 implies  $\mathcal{K}', w_0 \Vdash E$ . Moreover, it is not difficult to observe that  $\mathcal{K}', w_0 \not\vdash E_i$  for every  $i \in I \cup J$ . Hence  $\mathcal{K}', w_0 \Vdash B$  and  $\mathcal{K}', w_0 \not\vdash C$ . Thus  $\mathcal{K}', w_0 \not\vdash E \rightarrow (B \rightarrow C)$ . Since  $\mathcal{M}(\mathbb{T})$  is assumed to have intuitionistic submodel property, we may conclude  $\mathcal{K}' \in \mathcal{M}(\mathbb{T})$  and hence  $\mathbb{T} \not\vdash E \rightarrow (B \rightarrow C)$ .  $\square$

**Theorem 2.15.**  $\Vdash_{\Gamma}^{\top} = \Vdash_{\Gamma^{\vee}}^{\top}$  and  $\Vdash_{\Gamma}^{\top} = \Vdash_{\Gamma^{\vee}}^{\top}$ .

*Proof.* We only show  $A \Vdash_{\Gamma}^{\top} B$  iff  $A \Vdash_{\Gamma^{\vee}}^{\top} B$  and leave similar arguments for other statements to the reader. Right-to-left direction holds since  $\Gamma \subseteq \Gamma^{\vee}$ . For other direction assume that  $A \Vdash_{\Gamma}^{\top} B$  and let  $E \in \Gamma^{\vee}$  such that  $\mathbb{T} \vdash E \rightarrow A$ . Then  $E = \bigvee_i E_i$  with  $E_i \in \Gamma$ . Hence for every  $i$  we have  $\mathbb{T} \vdash E_i \rightarrow A$ . Then  $A \Vdash_{\Gamma}^{\top} B$  implies  $\mathbb{T} \vdash E_i \rightarrow B$ . Thus  $\mathbb{T} \vdash E \rightarrow B$ , as desired.  $\square$

## 2.11 Greatest lower bounds

Given a set  $\Gamma \cup \{A\} \subseteq \mathcal{L}_{\square}$  and a logic  $\mathbb{T}$ , we say that  $B$  is a  $(\Gamma, \mathbb{T})$ -lower bound for  $A$  if:

1.  $B \in \Gamma$ ,
2.  $\mathbb{T} \vdash B \rightarrow A$ .

Moreover we say that  $B$  is the  $(\Gamma, \mathbb{T})$ -greatest lower bound ( $(\Gamma, \mathbb{T})$ -glb) for  $A$ , if for every  $(\Gamma, \mathbb{T})$ -lower bound  $B'$  for  $A$  we have  $\mathbb{T} \vdash B' \rightarrow B$ . Note that up to  $\mathbb{T}$ -provable equivalence relation, such glb is unique and we annotate it as  $\lfloor A \rfloor_{\Gamma}^{\top}$ .

We say that  $(\Gamma, \mathbb{T})$  is downward compact, if every  $A \in \mathcal{L}_{\square}$  has a  $(\Gamma, \mathbb{T})$ -glb  $\lfloor A \rfloor_{\Gamma}^{\top}$ . If  $\lfloor A \rfloor_{\Gamma}^{\top}$  can be effectively computed, we say that  $(\Gamma, \mathbb{T})$  is recursively downward compact. A main result in [Visser, 2002] states that  $(\text{NNIL}, \text{IPC})$  is recursively downward compact (see section 7 in [Visser, 2002]). We say that  $(\Gamma, \mathbb{T})$  is (recursively) strong downward compact, if it is (recursively) downward compact and for every  $\square B \in \text{sub}(\lfloor A \rfloor_{\Gamma}^{\top})$  either we have  $\square B \in \text{sub}(A)$  or  $B \in \Gamma\text{-NF}_0$ . We also inductively define  $\llbracket A \rrbracket_{\Gamma}^{\top}$ :

- $\llbracket a \rrbracket_\Gamma^\top = a$  for every atomic  $a$ .
- $\llbracket \cdot \rrbracket_\Gamma^\top$  commutes with  $\{\vee, \wedge, \rightarrow\}$ .
- $\llbracket \Box A \rrbracket_\Gamma^\top := \Box \llbracket A \rrbracket_\Gamma^\top$ .

**Question 1.** *One may similarly define the notion of least upper bounds and upward compactness. Does downward compactness imply upward compactness?*

**Theorem 2.16.**  *$B$  is the  $(\Gamma, \top)$ -glb for  $A$  iff*

- $B \in \Gamma$ ,
- $\top \vdash B \rightarrow A$ ,
- $A \overset{\top}{\approx} B$ .

Hence we have  $A \overset{\top}{\approx} \llbracket A \rrbracket_\Gamma^\top$ .

*Proof.* Left to the reader. □

**Question 2.** *As we saw in theorem 2.16, the glb may be expressed via preservativity relation  $\overset{\top}{\approx}$ . One may think of its variant which best suites for lub's:*

$$A \overset{*}{\approx} B \quad \text{iff} \quad \forall E \in \Gamma (\top \vdash A \rightarrow E \Rightarrow \top \vdash B \rightarrow E).$$

*We ask for an axiomatization for  $\overset{*}{\approx}$  when we let  $\top = \text{IPC}$  and  $\Gamma = \text{NNIL}$ .*

**Corollary 2.17.** *If  $\llbracket A \rrbracket_\Gamma^\top$  exists, then for every  $B \in \mathcal{L}_\Box$  we have*

$$\top \vdash \llbracket A \rrbracket_\Gamma^\top \rightarrow B \quad \text{iff} \quad A \overset{\top}{\approx} B.$$

*Proof.* First assume that  $\top \vdash \llbracket A \rrbracket_\Gamma^\top$ . Also let  $E \in \Gamma$  such that  $\top \vdash E \rightarrow A$ . Theorem 2.16 implies  $A \overset{\top}{\approx} \llbracket A \rrbracket_\Gamma^\top$  and hence  $\top \vdash E \rightarrow \llbracket A \rrbracket_\Gamma^\top$ . Then by  $\top \vdash \llbracket A \rrbracket_\Gamma^\top \rightarrow B$  we get  $\top \vdash E \rightarrow B$ , as desired.

For the other direction let  $A \overset{\top}{\approx} B$ . By definition we have  $\llbracket A \rrbracket_\Gamma^\top \in \Gamma$  and  $\top \vdash \llbracket A \rrbracket_\Gamma^\top \rightarrow A$ . Hence by  $A \overset{\top}{\approx} B$  we get  $\top \vdash \llbracket A \rrbracket_\Gamma^\top \rightarrow B$ , as desired. □

**Lemma 2.18.** *If  $(\Gamma, \top)$  is strong downward compact and  $\top \supseteq \text{iK4}$ , then for every  $A \in \mathcal{L}_\Box$  we have  $\llbracket A \rrbracket_\Gamma^\top \in \Gamma\text{-NF}$  and  $\text{TH}(\Gamma, \top) \vdash A \leftrightarrow \llbracket A \rrbracket_\Gamma^\top$ .*

*Proof.* Easy induction on the complexity of  $A$  and left to the reader. □

## 2.12 Modal logics with binary modal operator

A modal logic with a binary modal operator has been studied in the literature of provability logic at least for two intended meanings for  $A \triangleright B$ : (1)  $\top \vdash B$  is interpretable in  $\top \vdash A$  [Visser, 1990; Berarducci, 1990; de Jongh and Veltman, 1990; Goris and Joosten, 2011] and (2)  $\top \vdash E \rightarrow A$  implies  $\top \vdash E \rightarrow B$  for every  $\Sigma_1$  sentence  $E$  [Iemhoff, 2003; Iemhoff et al., 2005]. The first is considered for classical theories like  $\top$  and the second is considered for the intuitionistic theories like  $\text{HA}$ . [Iemhoff, 2001c, 2003] introduces the logic  $\text{iPH}$  (as will be defined in this section) over the language  $\mathcal{L}_\triangleright$ . Albert Visser and Dick de Jongh prove [Iemhoff, 2003] that  $\text{iPH}$  is sound for arithmetical interpretations in  $\text{HA}$  and conjecture that  $\text{iPH}$  is also complete for such interpretations. We also conjecture that  $\text{iPH}_\sigma$  (as will be defined in this subsection) is the  $\Sigma_1$ -preservativity logic of  $\text{HA}$  for  $\Sigma_1$ -substitutions, i.e.  $\text{iPH}_\sigma \vdash A$  iff for every  $\Sigma_1$ -substitution  $\sigma$  we have  $\text{HA} \vdash \sigma_{\text{HA}}(A)$  (see section 2.3).

Define  $\{\{\top, \Delta\}\}$  as a logic in the language  $\mathcal{L}_\triangleright$  with following axioms and rules:

## Axioms

T: All theorems of T.

$\forall(\Delta)$ :  $B \rightarrow C \triangleright \bigvee_{i=1}^{n+m} \{B\}_\Delta(E_i)$ , in which  $B = \bigwedge_{i=1}^n (E_i \rightarrow F_i)$  and  $C = \bigvee_{i=n+1}^{n+m} E_i$ .

Mont( $\Delta$ ):  $A \triangleright B \rightarrow (C \rightarrow A) \triangleright (C \rightarrow B)$  for every  $C \in \Delta$ .

Le:  $A \triangleright \Box A$  for every  $A$ .

Disj:  $(B \triangleright A \wedge C \triangleright A) \rightarrow (B \vee C) \triangleright A$ .

Conj:  $[(A \triangleright B) \wedge (A \triangleright C)] \rightarrow (A \triangleright (B \wedge C))$ .

Cut:  $[(A \triangleright B) \wedge (B \triangleright C)] \rightarrow (A \triangleright C)$ .

## Rules

MP:  $A, A \rightarrow B / B$ .

PNec:  $A \rightarrow B / A \triangleright B$ .

Also define  $\{\{\mathsf{T}, \Delta\}\}^+$  as  $\{\{\mathsf{T}, \Delta\}\}$  plus the following axiom:

4p:  $(B \triangleright C) \rightarrow \Box(B \triangleright C)$ .

Then define  $\mathsf{iPH} := \{\{\mathsf{iGL}, \mathsf{parb}\}\}$  and  $\mathsf{iPH}^+ := \{\{\mathsf{iGL}, \mathsf{parb}\}\}^+$ . Finally define  $\mathsf{iPH}_\sigma := \{\{\mathsf{iGLC}_a, \mathsf{atomb}\}\}$ .

**Remark 2.19.** *The Löb's preservativity principle  $(\Box A \rightarrow A) \triangleright A$ , is derivable in  $\mathsf{iPH}$ . This axiom was listed in the original axiomatization of  $\mathsf{iPH}$  in [Iemhoff, 2003; Iemhoff et al., 2005].*

*Proof.* Let us reason inside  $\mathsf{iPH}$ . By Le we have  $(\Box A \rightarrow A) \triangleright \Box(\Box A \rightarrow A)$ . Also by Löb's axiom in  $\mathsf{iGL}$  and necessitation we get  $\Box(\Box A \rightarrow A) \triangleright \Box A$ . Thus by Cut we have  $(\Box A \rightarrow A) \triangleright \Box A$ . Since  $(\Box A \rightarrow A) \triangleright (\Box A \rightarrow A)$  by conjunction axiom we have  $(\Box A \rightarrow A) \triangleright (\Box A \wedge (\Box A \rightarrow A))$ . By necessitation we have  $(\Box A \wedge (\Box A \rightarrow A)) \triangleright A$  and thus Cut implies  $(\Box A \rightarrow A) \triangleright A$ .  $\square$

**Theorem 2.20.**  *$\mathsf{iPH}^+$  is sound for strong arithmetical interpretations in HA, i.e. for every arithmetical substitution  $\alpha$  and  $A \in \mathcal{L}_\triangleright$  with  $\mathsf{iPH}^+ \vdash A$  we have  $\mathsf{HA} \vdash \alpha_{\mathsf{HA}}^+(A)$  (see section 2.3).*

*Proof.* [Iemhoff, 2001c, 2003] proves that  $\mathsf{iPH} \vdash A$  implies  $\mathsf{HA} \vdash \alpha_{\mathsf{HA}}(A)$ . The same proof works for  $\mathsf{iPH}$  and  $\alpha_{\mathsf{HA}}^+(A)$  as well. Also the validity of 4p for strong interpretations is obvious.  $\square$

## 2.13 Simultaneous fixed-point theorem

It is well-known that the Gödel-Löb's logic GL proves the fixed-point theorem, i.e. for every  $A \in \mathcal{L}_\Box$  and atomic  $a$  such that  $a$  only occurs inside the scope of boxes, there is a unique (up to GL-provable equivalence relation) fixed point for  $A$  with respect to  $a$ , i.e. there is a proposition  $D$  such that

$$\mathsf{GL} \vdash A[a : D] \leftrightarrow D$$

Moreover one may consider  $D$  such that it only contains atomics appeared in  $A$  other than  $a$ . A well-known extension of this result is the simultaneous fixed-point theorem, which we state its intuitionistic version in the following theorem.

As we will see in this paper, we mainly deal with outer substitutions  $\hat{\theta}$ , i.e. we do not substitute variables which are inside scope of boxes. The fixed-point theorem helps us (in the proof of theorem 5.4 which is major step in the arithmetical completeness of  $\mathsf{iGLH}$ ) to convert usual substitutions to outer substitutions.

[Iemhoff et al., 2005] proves the fixed-point theorem for  $\mathsf{iGL}$ . In the following theorem we extend it to a simultaneous version, just like in the classical case.

**Theorem 2.21.** Let  $\vec{E} := \{E_1, \dots, E_m\}$  and  $\vec{a} = \{a_1, \dots, a_m\}$  such that every occurrences of  $a_i$  in  $E_j$  is in the scope of some  $\Box$ . Then there is a substitution  $\tau$  which is the simultaneous fixed point of  $\vec{a}$  with respect to  $\vec{E}$  in  $\text{iGL}$ , i.e.

- $\text{iGL} \vdash \tau(E_i) \leftrightarrow \tau(a_i)$  for every  $1 \leq i \leq m$ .
- $\tau$  is identity on every atomic  $a \notin \vec{a}$ .
- $\text{sub}^{\text{atom}}(\tau(a_i)) \subseteq (\text{sub}^{\text{atom}}(\vec{E}) \setminus \vec{a})$ .

*Proof.* The classical syntactical proof for this result with  $n = 1$  is valid for its intuitionistic counterpart. We refer the reader to [Smoryński, 1985, theorem 1.3.5] or [Boolos, 1995, section 8]. Then we may use induction on  $m$  and prove the general case as follows. As induction hypothesis, assume that we already have the statement of this theorem for  $m$  and prove it for  $m + 1$ . So assume that  $\vec{E} := \{E_1, \dots, E_m, E_{m+1}\}$  and  $\vec{a} := \{a_1, \dots, a_m, a_{m+1}\}$  are given. By induction hypothesis there is some substitution  $\tau'$  such that

- $\text{iGL} \vdash \tau'(E_i) \leftrightarrow \tau'(a_i)$  for every  $1 \leq i \leq m$ .
- $\tau'$  is identity on every atomic  $a \notin (\vec{a} \setminus a_{m+1})$ .
- $\text{sub}^{\text{atom}}(\tau'(a_i)) \subseteq \{a_{m+1}\} \cup (\text{sub}^{\text{atom}}(\vec{E}) \setminus \vec{a})$  for  $1 \leq i \leq m$ .

Then there is a fixed-point  $D$  for  $\tau'(E_{m+1})$  with respect to  $a_{m+1}$ , i.e.:

$$\text{iGL} \vdash D \leftrightarrow \tau'(E_{m+1})[a_{m+1} : D].$$

Finally define  $\tau$  as follows:

$$\tau(a) := \begin{cases} (\tau'(a))[a_{m+1} : D] & : a = a_i \text{ for } 1 \leq i \leq m \\ D & : a = a_{m+1} \\ a & : \text{otherwise} \end{cases}.$$

Then it is not difficult to observe that  $\tau$  satisfies all required conditions for the simultaneous fixed-point.  $\square$

## 2.14 Two crucial results

There are two results which are crucial for arithmetical completeness of the provability logic of HA: (1) the characterization of  $\Sigma_1$ -provability logic of HA (theorem 2.22) and (2) theorem 2.23 which implies the characterization of  $\mathbb{K}_{\Gamma}^{\text{IPC}}$  and  $\mathbb{K}_{\Delta}^{\text{IPC}}$  for  $\Gamma := \text{NNIL}(\text{par})$  and  $\Delta := \downarrow^{\text{IPC}} \Gamma$  (corollary 2.24).

**Theorem 2.22.** The  $\Sigma_1$ -Provability logic of HA is  $\text{iGLC}_a\text{H}_{\sigma}$ , i.e.  $\text{iGLC}_a\text{H}_{\sigma} \vdash A$  iff for every  $\Sigma_1$ -substitution  $\alpha$  we have  $\text{HA} \vdash \alpha_{\text{HA}}(A)$ .

*Proof.* [Ardeshir and Mojtahedi, 2018].  $\square$

**Theorem 2.23.** Given  $A \in \mathcal{L}_0$ , there is a finite set  $\Pi \subseteq \downarrow\text{N}(\text{sub}^{\text{par}}(A))$  such that

1.  $\text{IPC} \vdash \bigvee \Pi \rightarrow A$ .
2.  $\llbracket \text{IPC}, \text{par} \rrbracket \vdash A \triangleright \bigvee \Pi$  (for definition of  $\llbracket \text{IPC}, \text{par} \rrbracket$  see section 2.10).
3.  $\Pi$  is computable as a function of  $A$ . Moreover for every  $D \in \Pi$ , the substitution  $\theta$  with  $D \xrightarrow[\text{IPC}]{\theta} D^{\dagger} \in \text{N}(\text{sub}^{\text{par}}(A))$  is computable.
4.  $c_{\rightarrow}(B) \leq c_{\rightarrow}(A) + 1 + \#\text{sub}^{\text{par}}(A)$  for every  $B \in \Pi$ .



5.  $\text{sub}^{\text{atom}}(B) \subseteq \text{sub}^{\text{atom}}(A)$  for every  $B \in \Pi$ .

*Proof.* Theorems 3.12, 3.27 and 4.15 from [Mojtahedi, 2022a].  $\square$

**Corollary 2.24.** Let  $\mathbb{T} := \text{IPC}$  and  $\Gamma := \text{NNIL}(\text{par})$  and  $\Delta := \downarrow^{\text{IPC}} \Gamma$ . Then for every  $A, B \in \mathcal{L}_0$

$$A \vDash_{\Delta}^{\mathbb{T}} B \quad \text{iff} \quad A \vDash_{\Gamma}^{\mathbb{T}} B \quad \text{iff} \quad \llbracket \text{IPC, par} \rrbracket \vdash A \triangleright B.$$

*Proof.*  $\llbracket \text{IPC, par} \rrbracket \vdash A \triangleright B$  implies  $A \vDash_{\Gamma}^{\mathbb{T}} B$ : One must use induction on the complexity of proof  $\llbracket \text{IPC, par} \rrbracket \vdash A \triangleright B$ . All cases are easy except for Disj and  $\mathbb{V}_{\text{AR}}^{\text{par}}$ , for which we refer the reader to lemma 4.5 in [Mojtahedi, 2022a]. A similar reasoning for modal logics presented in theorem 3.39 in this paper.

$A \vDash_{\Gamma}^{\mathbb{T}} B$  implies  $A \vDash_{\Delta}^{\mathbb{T}} B$ : Lemma 2.10.

$A \vDash_{\Delta}^{\mathbb{T}} B$  implies  $\llbracket \text{IPC, par} \rrbracket \vdash A \triangleright B$ : Let  $A \vDash_{\Delta}^{\mathbb{T}} B$  and  $\Pi$  be a set of propositions as provided by theorem 2.23. Then for every  $E \in \Pi$  we have  $\text{IPC} \vdash E \rightarrow A$  and since  $\Pi \subseteq \Delta$  we get  $\text{IPC} \vdash E \rightarrow B$ . Thus  $\text{IPC} \vdash \bigvee \Pi \rightarrow B$  and  $\llbracket \text{IPC, par} \rrbracket \vdash \bigvee \Pi \triangleright B$ . Since  $\llbracket \text{IPC, par} \rrbracket \vdash A \triangleright \bigvee \Pi$  Cut implies  $\llbracket \text{IPC, par} \rrbracket \vdash A \triangleright B$ .  $\square$

### 3 Preservativity and relative admissibility

In this section, we study preservativity  $\vDash_{\Gamma}^{\mathbb{T}}$  and relative admissibility  $\vDash_{\Gamma}^{\mathbb{T}}$  for several pairs  $(\Gamma, \mathbb{T})$  of sets  $\Gamma \subseteq \mathcal{L}_{\square}$  and intuitionistic modal logics  $\mathbb{T}$ . The main application is to use them for the axiomatization of provability logic of HA and prove its decidability. More precisely we have the following results in this section. The main result in section 3.1 is theorem 3.2 in which we show that NNIL propositions can equivalently represented as disjunctions of prime and NNIL propositions. In section 3.2 we define relative projectivity in the modal language. In the rest of subsections in this section we will characterize the following relations:

- Section 3.3:  $\llbracket \mathbb{T}, \text{atomb} \rrbracket \text{Le} = \vDash_{\text{SN}}^{\mathbb{T}} = \vDash_{\text{SN}^{\vee}}^{\mathbb{T}} = \vDash_{\text{SPN}}^{\mathbb{T}} = \vDash_{\text{SPN}^{\vee}}^{\mathbb{T}}$ , whenever  $\mathbb{T}$  is TYPE- $\sigma$ .
- Section 3.4:  $\llbracket \mathbb{T}, \text{parb} \rrbracket = \vDash_{\downarrow \text{N}(\square)}^{\mathbb{T}} = \vDash_{\downarrow \text{N}(\square)^{\vee}}^{\mathbb{T}} = \vDash_{\downarrow \text{PN}(\square)}^{\mathbb{T}} = \vDash_{\downarrow \text{PN}(\square)^{\vee}}^{\mathbb{T}} = \vDash_{\text{N}(\square)}^{\mathbb{T}} = \vDash_{\text{N}(\square)^{\vee}}^{\mathbb{T}} = \vDash_{\text{PN}(\square)}^{\mathbb{T}} = \vDash_{\text{PN}(\square)^{\vee}}^{\mathbb{T}}$ , whenever  $\mathbb{T}$  is TYPE- $\alpha$ .
- Section 3.5:  $\llbracket \mathbb{T}, \text{parb} \rrbracket \text{Le}^{-} = \vDash_{\downarrow \text{SN}(\square)}^{\mathbb{T}} = \vDash_{\downarrow \text{SN}(\square)^{\vee}}^{\mathbb{T}} = \vDash_{\downarrow \text{SPN}(\square)}^{\mathbb{T}} = \vDash_{\downarrow \text{SPN}(\square)^{\vee}}^{\mathbb{T}} = \vDash_{\text{SN}(\square)}^{\mathbb{T}} = \vDash_{\text{SN}(\square)^{\vee}}^{\mathbb{T}} = \vDash_{\text{SPN}(\square)}^{\mathbb{T}} = \vDash_{\text{SPN}(\square)^{\vee}}^{\mathbb{T}}$ , whenever  $\mathbb{T}$  is TYPE- $\alpha$ .
- Section 3.6:  $\llbracket \mathbb{T}, \text{parb} \rrbracket \text{Le} = \vDash_{\downarrow \text{C,SN}(\square)}^{\mathbb{T}} = \vDash_{\downarrow \text{C,SN}(\square)^{\vee}}^{\mathbb{T}} = \vDash_{\downarrow \text{C,SPN}(\square)}^{\mathbb{T}} = \vDash_{\downarrow \text{C,SPN}(\square)^{\vee}}^{\mathbb{T}}$ , whenever  $\mathbb{T}$  is TYPE- $\alpha$ .
- Section 3.7:  $\llbracket \mathbb{T}, \text{parb} \rrbracket \text{LeA} = \vDash_{\text{SN}(\square)}^{\mathbb{T}} = \vDash_{\text{SPN}(\square)}^{\mathbb{T}} = \vDash_{\text{SPN}(\square)^{\vee}}^{\mathbb{T}}$ , whenever  $\mathbb{T}$  is TYPE- $\alpha$ .
- If  $\mathbb{T}$  is decidable, then all above mentioned relations are also decidable.

**Important convention:** All over this section we fix some set  $\mathbb{T} \supseteq \text{IPC}$  of modal propositions which is closed under modeus ponens and some class  $\mathcal{M}(\mathbb{T})$  of rooted Kripke models for which  $\mathbb{T}$  is sound and complete. Thus in the rest of this section we may omit the superscript  $\mathbb{T}$  from notations, e.g we may use shorter notations  $\mathbb{C}$ ,  $\mathbb{P}$  and  $\downarrow \Gamma$  instead of  $\mathbb{C}^{\mathbb{T}}$ ,  $\mathbb{P}^{\mathbb{T}}$  and  $\downarrow^{\mathbb{T}} \Gamma$ , respectively. Also we define three types of modal logics as follows.

- TYPE-0:  $\mathbb{T} \supseteq \text{iK4}$  and  $\mathcal{M}(\mathbb{T})$  has extension property (see section 3.1.2).
- TYPE- $\alpha$ :  $\mathbb{T} \supseteq \text{iK4}$  and  $\mathbb{T}$  is closed under necessitation and  $\mathcal{M}(\mathbb{T})$  has extension property (see section 3.1.2) and the intuitionistic submodel property (see section 2.6) and  $\mathbb{T}$  is closed under outer substitutions, i.e.  $A \in \mathbb{T}$  implies  $\hat{\theta}(A) \in \mathbb{T}$  for every substitution  $\theta$ .
- TYPE- $\sigma$ :  $\mathbb{T} \supseteq \text{iK4C}_a$  and  $\mathbb{T}$  is closed under necessitation and  $\mathcal{M}(\mathbb{T})$  has extension property (see section 3.1.2) and the intuitionistic submodel property (see section 2.6).

In the rest of this section, in the statements of theorems the notation (TYPE-0) means that we assumed  $\mathsf{T}$  satisfies conditions of TYPE-0. We have similar notation for TYPE- $\alpha$  and TYPE- $\sigma$ .

**Remark 3.1.**  $\mathsf{iK4}$  and  $\mathsf{iGL}$  are TYPE- $\alpha$ . Also  $\mathsf{iK4C}_a$  and  $\mathsf{iGLC}_a$  are TYPE- $\sigma$ . The finite model property for  $\mathsf{iK4}$ ,  $\mathsf{iGL}$ ,  $\mathsf{iK4C}_a$  and  $\mathsf{iGLC}_a$  implies their decidability.

*Proof.* Left to the reader.  $\square$

In the rest of this paper we may use above remark without mentioning.

### 3.1 Prime factorization for NNIL

In this section we prove that every NNIL proposition  $A \in \mathcal{L}_\square$  can be decomposed to a disjunction  $\bigvee_i A_i$  of  $\mathsf{T}$ -prime NNIL propositions (theorem 3.2). We take a route similar to the one we had in section 3.5 of [Mojtahedi, 2022a] for non-modal language. To show this, we will need two other equivalent notions:  $\mathsf{T}$ -component and  $\mathsf{T}$ -extendible. We first show that every  $A$  can be decomposed to  $\mathsf{T}$ -components (corollary 3.5), and then show that every  $\mathsf{T}$ -component is  $\mathsf{T}$ -extendible (lemma 3.6) and finally show that every  $\mathsf{T}$ -extendible is  $\mathsf{T}$ -prime (lemma 3.7). We will see in theorem 3.8 that these three notions are equivalent.

Remember that  $\mathsf{P}$  indicates the set of all  $\mathsf{T}$ -prime propositions, i.e.

$$\mathsf{P} := \{A \in \mathcal{L}_\square : \mathsf{T} \vdash A \rightarrow (B \vee C) \text{ implies } \mathsf{T} \vdash A \rightarrow B \text{ or } \mathsf{T} \vdash A \rightarrow C\}.$$

**Theorem 3.2.** (TYPE-0) *Up to  $\mathsf{iK4}$ -provable equivalence relation we have  $\mathsf{N}(\square) = \mathsf{PN}(\square)^\vee$  and  $\mathsf{SN}(\square) = \mathsf{SPN}(\square)^\vee$ . Also if  $\mathsf{T} \supseteq \mathsf{iK4C}_a$ , up to  $\mathsf{iK4C}_a$ -provable equivalence relation we have  $\mathsf{N} = \mathsf{PN}^\vee$  and  $\mathsf{SN} = \mathsf{SPN}^\vee$ .*

*Proof.* Direct consequence of corollary 3.5 and lemmas 3.6 and 3.7.  $\square$

**Corollary 3.3.** (TYPE-0)  $\mathbb{V}_{\mathsf{SN}(\square)}^\mathsf{T} = \mathbb{V}_{\mathsf{SPN}(\square)}^\mathsf{T}$  and  $\mathbb{V}_{\mathsf{SN}(\square)}^\mathsf{T} = \mathbb{V}_{\mathsf{SPN}(\square)}^\mathsf{T}$  and  $\mathbb{V}_{\mathsf{N}(\square)}^\mathsf{T} = \mathbb{V}_{\mathsf{PN}(\square)}^\mathsf{T}$  and if  $\mathsf{T} \supseteq \mathsf{iK4C}_a$  then  $\mathbb{V}_{\mathsf{SN}}^\mathsf{T} = \mathbb{V}_{\mathsf{SPN}}^\mathsf{T}$ .

*Proof.* Direct consequence of theorems 2.15 and 3.2.  $\square$

In the rest of this subsection we have technical lemmas needed for the proof of above theorem.

#### 3.1.1 $\mathsf{T}$ -components

Given  $A \in \mathsf{NNIL}$ , we say that  $A$  is a  $\mathsf{T}$ -component if  $A = \bigwedge \Gamma \wedge \bigwedge \Delta$  with the following properties:

- Every  $B \in \Gamma$  is atomic or boxed.
- Every  $B \in \Delta$  is an implication  $C \rightarrow D$  for some atomic or boxed  $C$  such that  $\mathsf{T} \not\vdash \bigwedge \Gamma \rightarrow C$ .

Let us define  $\mathsf{NNIL}^+ := \{A \in \mathsf{NNIL} : E \rightarrow F \in \mathsf{sub}_\circ(A) \text{ implies } E \in \mathsf{atomb}\}$ . In other words,  $\mathsf{NNIL}^+$  includes all NNIL propositions such that every antecedent occurring outside of boxes is either atomic or boxed. Obviously every  $A \in \mathsf{NNIL}$  ( $\mathsf{SNNIL}$ ) can be converted to some  $A' \in \mathsf{NNIL}^+$  ( $A' \in \mathsf{SNNIL}^+$ ) via derivability in IPC ( $\mathsf{iK4}$ ).

**Lemma 3.4.** *Given  $A \in \mathsf{NNIL}^+$  ( $A \in \mathsf{SNNIL}^+$ ), there is a finite set  $\Gamma_A \subseteq \mathsf{NNIL}^+$  ( $\Gamma_A \subseteq \mathsf{SNNIL}^+$ ) of  $\mathsf{T}$ -components such that  $\mathsf{T} \vdash A \leftrightarrow \bigvee \Gamma_A$  and  $\mathsf{sub}_\circ^{\mathsf{atomb}}(\Gamma_A) \subseteq \mathsf{sub}_\circ^{\mathsf{atomb}}(A)$ .*

*Proof.* We use induction on  $\mathsf{sub}_\circ^{\mathsf{atomb}}(A)$  (the set of atomics or boxed formulas occurring outside of the scope of boxes in  $A$ ) ordered by  $\subset$  and find some finite set  $\Gamma_A \subseteq \mathsf{NNIL}^+$  ( $\Gamma_A \subseteq \mathsf{SNNIL}^+$ ) of  $\mathsf{T}$ -components with  $\mathsf{sub}_\circ^{\mathsf{atomb}}(\Gamma_A) \subseteq \mathsf{sub}_\circ^{\mathsf{atomb}}(A)$  and  $\mathsf{T} \vdash \bigvee \Gamma_A \leftrightarrow A$ .

As induction hypothesis assume that for every  $\mathsf{T}' \supseteq \mathsf{IPC}$  and  $B \in \mathsf{NNIL}^+$  ( $B \in \mathsf{SNNIL}^+$ ) with  $\mathsf{sub}_\circ^{\mathsf{atomb}}(B) \subsetneq \mathsf{sub}_\circ^{\mathsf{atomb}}(A)$  there is a finite set  $\Gamma_B \subseteq \mathsf{NNIL}^+$  ( $\Gamma_B \subseteq \mathsf{SNNIL}^+$ ) of  $\mathsf{T}'$ -components

such that  $\mathsf{T}' \vdash B \leftrightarrow \bigvee \Gamma_B$  and  $\text{sub}_0^{\text{atomb}}(\Gamma_B) \subseteq \text{sub}_0^{\text{atomb}}(B)$ . For the induction step, assume that  $A \in \text{NNIL}^+$  ( $A \in \text{SNNIL}^+$ ) is given. Using derivability in IPC one may easily find finite sets  $\Gamma_i$  and  $\Delta_i$  such that

- $\text{IPC} \vdash A \leftrightarrow \bigvee_{i=1}^n A_i$ , in which  $A_i := \bigwedge \Gamma_i \wedge \bigwedge \Delta_i$ .
- $\Delta_i$  includes either atomic or boxed propositions.
- $\Gamma_i$  includes implications with atomic or boxed antecedents.
- $\text{sub}_0^{\text{atomb}}(A_i) \subseteq \text{sub}_0^{\text{atomb}}(A)$ .
- $A_i \in \text{NNIL}^+$  ( $A_i \in \text{SNNIL}^+$ ).

It is enough to decompose every  $A_i$  to  $\mathsf{T}$ -components. If  $\mathsf{T} \not\vdash \bigwedge \Delta_i \rightarrow E$  for every antecedent  $E$  of an implication in  $\Gamma_i$ , then  $A_i$  already is a  $\mathsf{T}$ -component and we are done. Otherwise, there is some  $E \rightarrow F \in \Gamma_i$  such that  $\mathsf{T} \vdash \bigwedge \Delta_i \rightarrow E$ . Then let  $A'_i := A_i[E : \top]$ , i.e. the replacement of every outer occurrences of  $E$  in  $A_i$  with  $\top$ . Also let  $\mathsf{T}' := \mathsf{T} + E$ . Hence  $\text{sub}_0^{\text{atomb}}(A'_i) \subsetneq \text{sub}_0^{\text{atomb}}(A)$  and by induction hypothesis we may decompose  $A'_i$  to  $\mathsf{T}'$ -components:

$$\mathsf{T}, E \vdash A'_i \leftrightarrow \bigvee_j B_j$$

It is not difficult to observe that if  $B_j$  is a  $\mathsf{T}'$ -component then  $B'_j := E \wedge B_j$  is a  $\mathsf{T}$ -component. Moreover  $\mathsf{T} \vdash E \wedge A'_i \leftrightarrow \bigvee_j B'_j$  and since  $\text{IPC} \vdash (E \wedge A'_i) \leftrightarrow (E \wedge A_i)$  and  $\mathsf{T} \vdash A_i \rightarrow E$ , we get

$$\mathsf{T} \vdash A_i \leftrightarrow \bigvee_j B'_j$$

Hence we have decomposed  $A_i$  to  $\mathsf{T}$ -components  $B'_j$  with  $\text{sub}_0^{\text{atomb}}(B'_j) \subseteq \text{sub}_0^{\text{atomb}}(A)$ .  $\square$

**Corollary 3.5.** *Every  $A \in \text{NNIL}$  can be decomposed to  $\mathsf{T}$ -components, i.e. there is a finite set of  $\mathsf{T}$ -components  $\Gamma_A$  such that  $\mathsf{T} \vdash A \leftrightarrow \bigvee \Gamma_A$ . Moreover:*

1. if  $A \in \text{N}(\square)$  then  $\Gamma_A \subseteq \text{N}(\square)$ ,
2. if  $A \in \text{SN}$  and  $\mathsf{T} \supseteq \text{iK4}$  then  $\Gamma_A \subseteq \text{SN}$ ,
3. if  $A \in \text{SN}(\square)$  and  $\mathsf{T} \supseteq \text{iK4}$  then  $\Gamma_A \subseteq \text{SN}(\square)$ .

*Proof.* Easy consequence of lemma 3.4 and left to reader. The condition  $\mathsf{T} \supseteq \text{iK4}$  is only needed to convert  $A \in \text{SNNIL}$  to some  $A' \in \text{SNNIL}^+$ .  $\square$

### 3.1.2 $\mathsf{T}$ -Extension property

Let  $\mathcal{H}$  and  $\mathcal{H}'$  be two sets of rooted Kripke models. In the following lines we are going to define a Kripke model  $\sum(\mathcal{H}, \mathcal{H}')$ . Roughly speaking it is resulted by adding a fresh root beneath  $\mathcal{H}$  having  $\square$ -access to the roots of  $\mathcal{H}'$ . More precisely define  $\sum(\mathcal{H}, \mathcal{H}') := (W, \preceq, \square, V)$  as follows. Assume that the sets of nodes of Kripke models in  $\mathcal{H} \cup \mathcal{H}'$  are disjoint. Moreover we assume that every  $\mathcal{K} \in \mathcal{H} \cup \mathcal{H}'$  is of the form  $\mathcal{K} := (W_{\mathcal{K}}, \preceq_{\mathcal{K}}, \square_{\mathcal{K}}, V_{\mathcal{K}})$ .

- $W := \{w_0\} \cup \bigcup_{\mathcal{K} \in \mathcal{H} \cup \mathcal{H}'} W_{\mathcal{K}}$ , in which  $w_0 \notin W_{\mathcal{K}}$  for every  $\mathcal{K} \in \mathcal{H} \cup \mathcal{H}'$ .
- $\preceq := (\bigcup_{\mathcal{K} \in \mathcal{H} \cup \mathcal{H}'} \preceq_{\mathcal{K}}) \cup \{(w_0, w) : \exists \mathcal{K} \in \mathcal{H}' (w_0^{\mathcal{K}} \preceq_{\mathcal{K}} w)\}$ , in which  $w_0^{\mathcal{K}}$  is the root of  $\mathcal{K}$ .
- $\square := (\bigcup_{\mathcal{K} \in \mathcal{H} \cup \mathcal{H}'} \square_{\mathcal{K}}) \cup \{(w_0, w) : \exists \mathcal{K} \in \mathcal{H}' (w_0^{\mathcal{K}} \square_{\mathcal{K}} w) \text{ or } \exists \mathcal{K} \in \mathcal{H} (w_0^{\mathcal{K}} \square_{\mathcal{K}} w)\}$ .
- $V := \bigcup_{\mathcal{K} \in \mathcal{H} \cup \mathcal{H}'} V_{\mathcal{K}}$ . Note that this means empty valuation for the fresh root  $w_0$ .

Two Kripke models are said to be *variant* of each other if they share the same rooted frame and their valuations are identical except at the root at which they may have different valuations.

A class  $\mathcal{M}$  of rooted Kripke models is said to have *extension property* if for every finite set  $\mathcal{X} \subseteq \mathcal{M}$  there is some finite set of rooted Kripke models  $\mathcal{X}'$  such that a variant of  $\sum(\mathcal{X}, \mathcal{X}')$  belongs to  $\mathcal{M}$ . A proposition  $A$  is said to have  $\mathcal{M}$ -extension property if  $\mathcal{M} \cap \text{Mod}(A)$  has extension property, in which  $\text{Mod}(A)$  is the class of all Kripke models of  $A$ .

**Lemma 3.6.** (TYPE-0) *Let  $A \in \mathcal{L}_\square$  be a  $\mathsf{T}$ -component such that  $\mathsf{T} \vdash a \rightarrow \square a$  for every  $a \in \text{sub}_0^{\text{atom}}(A)$ . Then  $A$  has  $\mathcal{M}(\mathsf{T})$ -extension property.*

*Proof.* Let  $\mathcal{X} \subseteq \mathcal{M}(\mathsf{T})$  be a finite set of Kripke models of a component  $A = \bigwedge \Gamma \wedge \bigwedge \Delta$ . Let

$$\Gamma = \{a_1, \dots, a_n, \square E_1, \dots, \square E_m\} \quad \text{and} \quad \Delta = \{b_1 \rightarrow G_1, \dots, b_k \rightarrow G_k, \square F_1 \rightarrow H_1, \dots, \square F_l \rightarrow H_l\}$$

in which  $a_i, b_j \in \text{atom}$ . For every  $i \leq l$  such that  $\mathcal{X} \not\models \square F_i$ , since  $\mathsf{T} \not\vdash \bigwedge \Gamma \rightarrow \square F_i$ , there is some  $\mathcal{K}'_i \in \mathcal{M}(\mathsf{T})$  such that  $\mathcal{K}'_i \Vdash \Gamma$  and  $\mathcal{K}'_i \not\models \square F_i$ . Thus there is some node  $w$  such that  $\mathcal{K}'_i, w \Vdash \Gamma'$  in which  $\Gamma' := \Gamma \cup \{E_1, \dots, E_m\}$  and  $\mathcal{K}'_i, w \not\models F_i$  (note that here we are using  $\mathcal{K}'_i \Vdash a_i \rightarrow \square a_i$ ). Define  $\mathcal{K}_i := (\mathcal{K}'_i)_w$ , i.e. the Kripke model generated by  $w$  in  $\mathcal{K}'_i$ . Let  $\mathcal{X}' := \{\mathcal{K}_1, \dots, \mathcal{K}_l\}$  and finally define  $\mathcal{K}$  as a variant of  $\sum(\mathcal{X}, \mathcal{X}')$  with the following valuation in the root:

$$\mathcal{K}, w_0 \Vdash a \quad \text{iff} \quad a \in \Gamma.$$

Then it is not difficult to observe that  $\mathcal{K}, w_0 \Vdash A$ . □

**Lemma 3.7.** *Every  $A \in \mathcal{L}_0$  with  $\mathcal{M}(\mathsf{T})$ -extension property, is  $\mathsf{T}$ -prime.*

*Proof.* Let  $\mathsf{T} \not\vdash A \rightarrow B$  and  $\mathsf{T} \not\vdash A \rightarrow C$ . Then there are Kripke models  $\mathcal{K}_1, \mathcal{K}_2 \in \mathcal{M}(\mathsf{T})$  such that  $\mathcal{K}_1, \mathcal{K}_2 \Vdash A$  and  $\mathcal{K}_1 \not\models B$  and  $\mathcal{K}_2 \not\models C$ . Since  $A$  has  $\mathcal{M}(\mathsf{T})$ -extension property, there is some Kripke model  $\mathcal{K} \in \mathcal{M}(\mathsf{T})$  which is an extension of  $\{\mathcal{K}_1, \mathcal{K}_2\}$  and  $\mathcal{K} \Vdash A$ . Since  $\mathcal{K}_1 \not\models B$ , we get  $\mathcal{K} \not\models B$ . Similarly  $\mathcal{K} \not\models C$ . Hence  $\mathcal{K} \not\models A \rightarrow (B \vee C)$  and thus  $\mathsf{T} \not\vdash A \rightarrow (B \vee C)$ . □

**Theorem 3.8.** (TYPE-0) *Let  $A \in \text{NNIL}$  with  $\mathsf{T} \vdash a \rightarrow \square a$  for every  $a \in \text{sub}_0^{\text{atom}}(A)$ . Following items are equivalent:*

1.  $A$  has  $\mathcal{M}(\mathsf{T})$ -extension property.
2.  $A$  is  $\mathsf{T}$ -prime.
3.  $A$  is a  $\mathsf{T}$ -component (up to  $\mathsf{T}$ -provable equivalency).

*Proof.* • 1  $\rightarrow$  2: lemma 3.7.

- 2  $\rightarrow$  3: Given  $A$ , first decompose it to  $\mathsf{T}$ -component and then use its  $\mathsf{T}$ -primality and deduce that  $A$  must be  $\mathsf{T}$ -equivalent to one of its  $\mathsf{T}$ -components.
- 3  $\rightarrow$  1: lemma 3.6. □

## 3.2 Relative projectivity for the modal language

In this section we extend the propositional notion of relative projectivity [Mojtahedi, 2022a] to the modal language.

Remember that  $\mathcal{L}_0(\text{parb})$  indicates the set of modal propositions with no occurrences of atomic variables outside  $\square$ . Let  $A \in \mathcal{L}_\square$  and  $\Gamma \subseteq \mathcal{L}_0(\text{parb})$ . A substitution  $\theta$  is called *A-projective* (in  $\mathsf{T}$ ) if

$$(3.1) \quad \text{For all atomic } a \text{ we have} \quad \mathsf{T} \vdash A \rightarrow (a \leftrightarrow \theta(a)).$$

A substitution  $\theta$ , is a  $\Gamma$ -fier for  $A \in \mathcal{L}_\square$ , if

$$\mathbb{T} \vdash \hat{\theta}(A) \in \Gamma \quad \text{i.e. } \hat{\theta}(A) \text{ is } \mathbb{T}\text{-equivalent to some } A' \in \Gamma.$$

In this case we use the notation  $A \xrightarrow[\mathbb{T}]{\theta} \Gamma$ . If  $\Gamma$  is a singleton  $\{A'\}$  we write  $A \xrightarrow[\mathbb{T}]{\theta} A'$  instead of  $A \xrightarrow[\mathbb{T}]{\theta} \{A'\}$ . The substitution  $\theta$  is a unifier for  $A$  if it is  $\{\mathbb{T}\}$ -fier for  $A$ . We say that a substitution  $\theta$  projects  $A$  to  $\Gamma$  in  $\mathbb{T}$  (notation:  $A \xrightarrow[\mathbb{T}]{\theta} \Gamma$ ) if  $\theta$  is  $A$ -projective in  $\mathbb{T}$  and  $A \xrightarrow[\mathbb{T}]{\theta} \Gamma$ . If  $\Gamma = \{A'\}$  we simplify  $A \xrightarrow[\mathbb{T}]{\theta} \{A'\}$  to  $A \xrightarrow[\mathbb{T}]{\theta} A'$ . We say that  $A$  is  $\Gamma$ -projective in  $\mathbb{T}$  (notation  $A \xrightarrow[\mathbb{T}]{} \Gamma$ ) if there is some  $\theta$  such that  $A \xrightarrow[\mathbb{T}]{\theta} \Gamma$ . Also  $\downarrow \Gamma$  indicates the set of all propositions which are  $\Gamma$ -projective in  $\mathbb{T}$ . Whenever  $\mathbb{T}$  can be inferred from the context, we may omit it and simply write  $\downarrow \Gamma$ . We say that  $A$  is projective, if it is  $\{\mathbb{T}\}$ -projective.

**Remark 3.9.** *If  $A \xrightarrow[\mathbb{T}]{} A' \in \mathcal{L}_0(\text{parb})$  then there is some  $\tau$  such which is identity on every atomic  $a \notin \text{sub}_o(A)$  and  $A \xrightarrow[\mathbb{T}]{\tau} A'$ .*

*Proof.* Let  $A \xrightarrow[\mathbb{T}]{\theta} A' \in \mathcal{L}_0(\text{parb})$  and define  $\tau$  as follows:

$$\tau(a) := \begin{cases} \theta(a) & : a \in \text{sub}_o(A) \\ a & : \text{otherwise} \end{cases}$$

Then obviously we have  $A \xrightarrow[\mathbb{T}]{\tau} A'$ . □

**Remark 3.10.** *Let  $\Gamma \subseteq \mathcal{L}_0(\text{parb})$  and  $\mathbb{T}$  be a modal logic closed under outer substitutions and includes IPC. Then for every  $\Gamma$ -projective proposition  $A$  in  $\mathbb{T}$ , there is a unique (modulo  $\mathbb{T}$ -provable equivalency)  $A^\dagger \in \Gamma$  such that  $A \xrightarrow[\mathbb{T}]{} A^\dagger$ . Such  $A^\dagger$  is called the  $\Gamma$ -projection of  $A$  in  $\mathbb{T}$ . Moreover  $\mathbb{T} \vdash A \rightarrow A^\dagger$ .*

*Proof.* Let  $A \xrightarrow[\mathbb{T}]{\theta} A'$  and  $A \xrightarrow[\mathbb{T}]{\tau} A''$  and  $A', A'' \in \Gamma$ . From the  $A$ -projectivity of  $\theta$  and  $\tau$ , for every atomic  $a$  we have  $A \vdash_{\mathbb{T}} \theta(a) \leftrightarrow \tau(a)$ . Hence  $A \vdash_{\mathbb{T}} \hat{\theta}(A) \leftrightarrow \hat{\tau}(A)$  and then  $A \vdash_{\mathbb{T}} A' \leftrightarrow A''$ . By applying  $\hat{\theta}$  to both sides of this derivation, we have  $\hat{\theta}(A) \vdash_{\mathbb{T}} \hat{\theta}(A') \leftrightarrow \hat{\theta}(A'')$ . Since  $A', A'' \in \mathcal{L}_0(\text{parb})$ ,  $\hat{\theta}(A') = A'$  and  $\hat{\theta}(A'') = A''$  and thus  $\vdash_{\mathbb{T}} A' \rightarrow A''$ . Similarly we have  $\vdash_{\mathbb{T}} A'' \rightarrow A'$ .

Next we show  $\mathbb{T} \vdash A \rightarrow A^\dagger$ . Let  $A \xrightarrow[\mathbb{T}]{\theta} A^\dagger$ . Then  $A \vdash_{\mathbb{T}} A \leftrightarrow \hat{\theta}(A)$  which implies  $A \vdash_{\mathbb{T}} A \leftrightarrow A^\dagger$  and hence  $\mathbb{T} \vdash A \rightarrow A^\dagger$ . □

### 3.3 SN-Preservativity

[Visser, 2002] axiomatizes the binary relation of NNIL-preservativity for the non-modal language and IPC (theorem 3.11). Here we will do a similar job for the modal language (theorem 3.13) and show that  $\llbracket \mathbb{T}, \text{atomb} \rrbracket \text{Le}$  axiomatizes  $\downarrow_{\text{SN}}^{\mathbb{T}}$  when  $\mathbb{T}$  is TYPE- $\sigma$ . Moreover we also show that  $(\text{SNNIL}, \mathbb{T})$  is recursively downward compact when  $\mathbb{T}$  is TYPE- $\sigma$ .

**Theorem 3.11.** *For every  $A, B \in \mathcal{L}_\square$ ,  $\llbracket \text{IPC}, \text{atomb} \rrbracket \text{Le} \vdash A \triangleright B$  iff  $A \downarrow_{\text{NNIL}}^{\text{IPC}} B$ . Moreover, for every  $A \in \mathcal{L}_\square$  there is some  $A^* \in \text{NNIL}$  which is effectively computable and  $\text{IPC} \vdash A^* \rightarrow A$  and  $\llbracket \text{IPC}, \text{atomb} \rrbracket \text{Le} \vdash A \triangleright A^*$  and moreover  $\text{sub}_o^{\text{atomb}}(A^*) \subseteq \text{sub}_o^{\text{atomb}}(A)$ .*

*Proof.* [Visser, 2002]. □

**Theorem 3.12.** (TYPE- $\sigma$ ) *Given  $A \in \mathcal{L}_\square$  we have  $A^h := (A^*)^\square = \lfloor A \rfloor_{\text{SN}}^{\mathbb{T}}$  and hence  $(\text{SN}, \mathbb{T})$  is recursively strong downward compact (see section 2.11).*

*Proof.* First we show that  $A^h := (A^*)^\square$  is the  $(\text{SN}, \text{T})$ -glb for  $A$ , in which  $A^*$  is as provided by theorem 3.11. By theorem 3.11,  $A^* \in \text{NNIL}$  and  $\text{IPC} \vdash A^* \rightarrow A$ . Also we have  $A^h = (A^*)^\square \in \text{SN}$  and  $\text{iK4C}_a \vdash A^h \rightarrow A^*$ . Lemma 3.15 implies  $\llbracket \text{T}, \text{atomb} \rrbracket \text{Le} \vdash A \triangleright A^h$  and then theorem 3.18 implies  $A \stackrel{\text{T}}{\underset{\text{SN}}{\approx}} A^h$ . Thus theorem 2.16 implies  $A^h = \lfloor A \rfloor_{\text{SN}}^{\text{T}}$ .

Theorem 3.11 implies that  $A^*$  is computable and hence  $\lfloor A \rfloor_{\text{SN}}^{\text{T}} = A^h$  is so. Next we reason for the strongness. Theorem 3.11 implies that  $\text{sub}_0^{\text{atomb}}(A^*) \subseteq \text{sub}_0^{\text{atomb}}(A)$ , and hence for every boxed subformula  $\square B$  of  $A^h := (A^*)^\square$  either we have  $\square B \in \text{sub}(A)$  or  $B \in \text{SN-NF}_0$ .  $\square$

**Theorem 3.13.** (TYPE- $\sigma$ )  $\llbracket \text{T}, \text{atomb} \rrbracket \text{Le}$  is sound and complete for SN-preservativity in  $\text{T}$ . More precisely, for every  $A, B \in \mathcal{L}_\square$ :

$$\llbracket \text{T}, \text{atomb} \rrbracket \text{Le} \vdash A \triangleright B \quad \text{iff} \quad A \stackrel{\text{T}}{\underset{\text{SN}}{\approx}} B.$$

*Proof.* The left-to-right direction (soundness) holds by theorem 3.18.

For the other direction, let  $A \stackrel{\text{T}}{\underset{\text{SN}}{\approx}} B$ . Theorem 3.12 and corollary 2.17 implies  $\text{T} \vdash A^h \rightarrow B$  and hence  $\llbracket \text{T}, \text{atomb} \rrbracket \text{Le} \vdash A^h \triangleright B$ . Also by lemma 3.15 we have  $\llbracket \text{T}, \text{atomb} \rrbracket \text{Le} \vdash A \triangleright A^h$  and then Cut implies desired result.  $\square$

**Corollary 3.14.** (TYPE- $\sigma$ )  $\text{T}$  is closed under  $(\cdot)^h$ , i.e. if  $\text{T} \vdash A$  then  $\text{T} \vdash A^h$ .

*Proof.* Let  $\text{T} \vdash A$  and hence  $\text{T} \vdash \text{T} \rightarrow A$ . Lemma 3.15 and remark 2.11 implies  $\llbracket \text{T}, \text{atomb} \rrbracket \text{Le} \vdash A \triangleright A^h$ . Then theorem 3.13 implies  $A \stackrel{\text{T}}{\underset{\text{SN}}{\approx}} A^h$  and hence  $\text{T} \vdash \text{T} \rightarrow A^h$ .  $\square$

**Lemma 3.15.** For every  $A \in \mathcal{L}_\square$  we have  $\llbracket \text{iK4C}_a, \text{atomb} \rrbracket \text{Le} \vdash A \triangleright A^h$ .

*Proof.* First note that by theorem 3.11 we have  $\llbracket \text{IPC}, \text{atomb} \rrbracket \text{Le} \vdash A \triangleright A^*$  and by remark 2.11  $\llbracket \text{iK4C}_a, \text{atomb} \rrbracket \text{Le} \vdash A \triangleright A^*$ . In the other hand lemma 3.16 implies  $\llbracket \text{iK4C}_a, \text{atomb} \rrbracket \text{Le} \vdash A^* \triangleright (A^*)^\square$ . Then Cut implies desired result.  $\square$

**Lemma 3.16.** For every  $A \in \text{NNIL}$ , we have  $\llbracket \text{iK4C}_a, \text{atomb} \rrbracket \text{Le} \vdash A \triangleright A^\square$ .

*Proof.* Use induction on the complexity of  $A$ :

- $A$  is atomic. Then  $A^\square = A \wedge \square A$  and by Le we have  $\llbracket \text{iK4C}_a, \text{atomb} \rrbracket \text{Le} \vdash A \triangleright \square A$ . Since  $\llbracket \text{iK4C}_a, \text{atomb} \rrbracket \text{Le} \vdash A \triangleright A$ , Conj implies the desired result.
- $A$  is boxed. Then  $A^\square = A$  and hence  $\text{iK4C}_a \vdash A \rightarrow A^\square$ . Thus  $\llbracket \text{iK4C}_a, \text{atomb} \rrbracket \text{Le} \vdash A \triangleright A^\square$ .
- $A = B \wedge C$ . By induction hypothesis,  $\llbracket \text{iK4C}_a, \text{atomb} \rrbracket \text{Le} \vdash B \triangleright B^\square$  and  $\llbracket \text{iK4C}_a, \text{atomb} \rrbracket \text{Le} \vdash C \triangleright C^\square$ . On the other hand,  $\text{T} \vdash (B \wedge C) \rightarrow B$  and  $\text{T} \vdash (B \wedge C) \rightarrow C$ . Hence  $\llbracket \text{iK4C}_a, \text{atomb} \rrbracket \text{Le} \vdash (B \wedge C) \triangleright B$  and  $\llbracket \text{iK4C}_a, \text{atomb} \rrbracket \text{Le} \vdash (B \wedge C) \triangleright C$ , which by Cut and Conj we have  $\llbracket \text{iK4C}_a, \text{atomb} \rrbracket \text{Le} \vdash (B \wedge C) \triangleright (B^\square \wedge C^\square)$ . Since  $(B \wedge C)^\square = B^\square \wedge C^\square$ , we have the desired result.
- $A = B \vee C$ . By induction hypothesis,  $\llbracket \text{iK4C}_a, \text{atomb} \rrbracket \text{Le} \vdash B \triangleright B^\square$  and  $\llbracket \text{iK4C}_a, \text{atomb} \rrbracket \text{Le} \vdash C \triangleright C^\square$ . On the other hand,  $\text{T} \vdash B \rightarrow (B \vee C)$  and  $\text{T} \vdash C \rightarrow (B \vee C)$ . Hence  $\llbracket \text{iK4C}_a, \text{atomb} \rrbracket \text{Le} \vdash B \triangleright (B \vee C)$  and  $\llbracket \text{iK4C}_a, \text{atomb} \rrbracket \text{Le} \vdash C \triangleright (B \vee C)$ , which by Disj and Cut we have  $\llbracket \text{iK4C}_a, \text{atomb} \rrbracket \text{Le} \vdash (B \vee C) \triangleright (B^\square \vee C^\square)$ . Since  $(B \vee C)^\square = B^\square \vee C^\square$ , we have the desired result.
- $A = B \rightarrow C$ . By induction hypothesis we have  $\llbracket \text{iK4C}_a, \text{atomb} \rrbracket \text{Le} \vdash C \triangleright C^\square$ . Since  $A \in \text{NNIL}$ , we have  $B \in \text{NI}$  and lemma 3.17 implies  $\llbracket \text{iK4C}_a, \text{atomb} \rrbracket \text{Le} \vdash (B \rightarrow C) \triangleright (B \rightarrow C^\square)$ . Since  $\text{iK4C}_a \vdash B \leftrightarrow B^\square$ , we get  $\llbracket \text{iK4C}_a, \text{atomb} \rrbracket \text{Le} \vdash (B \rightarrow C^\square) \triangleright (B^\square \rightarrow C^\square)$ . Finally by Le we get  $\llbracket \text{iK4C}_a, \text{atomb} \rrbracket \text{Le} \vdash (B^\square \rightarrow C^\square) \triangleright \square(B^\square \rightarrow C^\square)$  and hence  $\llbracket \text{iK4C}_a, \text{atomb} \rrbracket \text{Le} \vdash (B \rightarrow C) \triangleright (B \rightarrow C)^\square$ , as desired.  $\square$

**Lemma 3.17.** The following rule is admissible to  $\llbracket \text{iK4C}_a, \text{atomb} \rrbracket \text{Le}$ :  $\frac{A \triangleright B}{E \rightarrow A \triangleright E \rightarrow B} E \in \text{NI}$ .

*Proof.* Use induction on the complexity of  $E$ .

- If  $E$  is atomic or boxed, then it is same as the Montagna's rule.
- $E = E_1 \wedge E_2$ . Let  $\llbracket \text{iK4C}_a, \text{atomb} \rrbracket \text{Le} \vdash A \triangleright B$ . Then by induction hypothesis we have  $\llbracket \text{iK4C}_a, \text{atomb} \rrbracket \text{Le} \vdash (E_2 \rightarrow A) \triangleright (E_2 \rightarrow B)$ . Again by induction hypothesis we have  $\llbracket \text{iK4C}_a, \text{atomb} \rrbracket \text{Le} \vdash (E_1 \rightarrow (E_2 \rightarrow A)) \triangleright (E_1 \rightarrow (E_2 \rightarrow B))$  and thus by Cut and  $\text{iK4C}_a$  we get  $\llbracket \text{iK4C}_a, \text{atomb} \rrbracket \text{Le} \vdash E \rightarrow A \triangleright E \rightarrow B$ .
- $E = E_1 \vee E_2$ . Let  $\llbracket \text{iK4C}_a, \text{atomb} \rrbracket \text{Le} \vdash A \triangleright B$ . Then by induction hypothesis we have  $\llbracket \text{iK4C}_a, \text{atomb} \rrbracket \text{Le} \vdash (E_1 \rightarrow A) \triangleright (E_1 \rightarrow B)$  and  $\llbracket \text{iK4C}_a, \text{atomb} \rrbracket \text{Le} \vdash (E_2 \rightarrow A) \triangleright (E_2 \rightarrow B)$ . Hence  $\llbracket \text{iK4C}_a, \text{atomb} \rrbracket \text{Le} \vdash ((E_1 \rightarrow A) \wedge (E_2 \rightarrow A)) \triangleright ((E_1 \rightarrow B) \wedge (E_2 \rightarrow B))$ . Thus  $\llbracket \text{iK4C}_a, \text{atomb} \rrbracket \text{Le} \vdash E \rightarrow A \triangleright E \rightarrow B$ , as desired.  $\square$

**Theorem 3.18 (Soundness).**  $(\text{TYPE-}\sigma) \llbracket \text{T}, \text{atomb} \rrbracket \text{Le} \vdash A \triangleright B$  implies  $A \stackrel{\text{T}}{\approx}_{\text{SN}} B$ .

*Proof.* Let  $\llbracket \text{T}, \text{atomb} \rrbracket \text{Le} \vdash A \triangleright B$ . Theorem 2.12 implies  $A \stackrel{\text{T}}{\approx}_{\text{SPN}} B$  and then corollary 3.3 implies  $A \stackrel{\text{T}}{\approx}_{\text{SN}} B$ .  $\square$

In the following corollary, we consider  $\llbracket \text{T}, \text{atomb} \rrbracket \text{Le}$  as a binary relation which it axiomatizes.

**Corollary 3.19.**  $(\text{TYPE-}\sigma)$  The following equalities hold. Moreover if  $\text{T}$  is decidable, all above relations are decidable.

$$\llbracket \text{T}, \text{atomb} \rrbracket \text{Le} = \stackrel{\text{T}}{\approx}_{\text{SN}} = \stackrel{\text{T}}{\approx}_{\text{SN}^\vee} = \stackrel{\text{T}}{\approx}_{\text{SPN}} = \stackrel{\text{T}}{\approx}_{\text{SPN}^\vee}$$

*Proof.* Theorems 2.15 and 3.13 and corollary 3.3 imply equalities. For the decidability of  $\stackrel{\text{T}}{\approx}_{\text{SN}}$  we have following argument. Theorem 3.12 implies that  $\llbracket A \rrbracket_{\text{SN}}^{\text{T}}$  exists and is computable. Then by corollary 2.17 it is enough to decide  $\text{T} \vdash \llbracket A \rrbracket_{\text{SN}}^{\text{T}} \rightarrow B$ , which is provided by decidability of  $\text{T}$ .  $\square$

### 3.4 $\downarrow \text{N}(\square)$ -Preservativity and $\text{N}(\square)$ -Admissibility

In this section we show that  $\llbracket \text{T}, \text{parb} \rrbracket$  axiomatizes  $\stackrel{\text{T}}{\approx}_{\downarrow \text{N}(\square)}$  and  $\stackrel{\text{T}}{\approx}_{\text{N}(\square)}$  whenever  $\text{T}$  is  $\text{TYPE-}\alpha$ . Moreover we show that  $(\downarrow \text{N}(\square)^\vee, \text{T})$  is recursively strong downward compact whenever  $\text{T}$  is  $\text{TYPE-}\alpha$ .

**Lemma 3.20.** Let  $\alpha$  be a general substitution such that for every  $p \in \text{par}$  we have  $\alpha(p) \in \text{parb}$ . Then for every  $A, B \in \mathcal{L}_0$ :

$$\llbracket \text{IPC}, \text{par} \rrbracket \vdash A \triangleright B \quad \text{implies} \quad \llbracket \text{IPC}, \text{parb} \rrbracket \vdash \alpha(A) \triangleright \alpha(B).$$

*Proof.* Easy induction on the complexity of proof  $\llbracket \text{IPC}, \text{par} \rrbracket \vdash A \triangleright B$  and left to the reader.  $\square$

**Definition 3.21.** For every  $A \in \mathcal{L}_\square$  we define  $A_0^{\text{p}}, A_1^{\text{p}} \in \mathcal{L}_\square$  and  $\Pi_A^0, \Pi_A^1 \subseteq \mathcal{L}_\square$  as follows. Let  $\vec{p} := p_1, \dots, p_n$  includes all parameters occurring in  $A$  and  $\text{sub}^{\text{B}}(A) = \{\square B_1, \square B_2, \dots, \square B_m\}$ . Also let  $\vec{q} := q_1, q_2, \dots, q_m$  be a list of fresh parameters, i.e.  $q_i \notin \vec{p}$  for every  $i$  and they are pairwise distinct. Let the substitution  $\alpha$  such that  $\alpha(q_i) = \square B_i$  and  $\alpha(a) = a$  for every other atomic  $a$ . Given  $B \in \mathcal{L}_0(\text{sub}^{\text{B}}(A), \vec{p}, \text{var})$ , there is a unique  $B_0 \in \mathcal{L}_0(\vec{q}, \vec{p}, \text{var})$  such that  $\alpha(B_0) = B$ . In the rest of this definition we use  $B_0$  as this unique proposition in  $\mathcal{L}_0(\vec{q}, \vec{p}, \text{var})$  for every  $B \in \mathcal{L}_0(\text{sub}^{\text{B}}(A), \vec{p}, \text{var})$ . By theorem 2.23 there is some finite set  $\Pi_A^0 \subseteq \downarrow \text{N}(\vec{p}, \vec{q})$  with the following properties:

P1:  $\text{IPC} \vdash D_0 \rightarrow A_0$  for every  $D_0 \in \Pi_A^0$ .

P2:  $\llbracket \text{IPC}, \text{par} \rrbracket \vdash A_0 \triangleright \bigvee \Pi_A^0$ .

P3:  $\Pi_A^0$  is computable function of  $A$ . Moreover for every  $D_0 \in \Pi_A^0$  the substitution  $\theta_0$  with  $D_0 \xrightarrow[\text{IPC}]{\theta_0} D_0^\dagger \in \text{N}(\vec{p}, \vec{q})$  can be effectively computed.

P4:  $c_{\rightarrow}(D_0) \leq c_{\rightarrow}^{\square}(A)$  for every  $D_0 \in \Pi_A^0$ .

P5: For every  $D_0 \in \Pi_A^0$ , we have  $\text{sub}^{\text{atom}}(D_0) \subseteq \text{sub}^{\text{atom}}(A_0) = \vec{p} \cup \text{sub}_o^{\text{atom}}(A)$ .

Hence for every  $D_0 \in \Pi_A^0$ , there is some substitution  $\theta_0$  such that  $D_0 \xrightarrow{\theta_0} D_0^{\dagger} \in \mathbf{N}(\vec{p}, \vec{q})$ . Thus for  $\theta := \alpha \circ \theta_0$  we have  $\alpha(D_0) = D \xrightarrow[\text{IPC}]{\theta} D^{\dagger} = \alpha(D_0^{\dagger}) \in \mathbf{N}(\square)$ . Then define

$$A_0^{\text{p}} := \bigvee \Pi_A^0 \quad \text{and} \quad \Pi_A^1 := \{\alpha(B) : B \in \Pi_A^0\} \quad \text{and} \quad A_1^{\text{p}} := \bigvee \Pi_A^1.$$

Note that by P3, one may effectively compute  $\Pi_A^0$ ,  $\Pi_A^1$  and  $A_1^{\text{p}}$ .

**Lemma 3.22.** *Given  $A \in \mathcal{L}_{\square}$  we have:*

1.  $\Pi_A^1 \subseteq \downarrow \mathbf{N}(\square)$  and hence  $A_1^{\text{p}} \in \downarrow \mathbf{N}(\square)^{\vee}$ .
2.  $\llbracket \text{IPC, parb} \rrbracket \vdash A \triangleright A_1^{\text{p}}$ .

*Proof.* 1. We show that for every  $D \in \Pi_A^1$  we have  $D \in \downarrow \mathbf{N}(\square)$ . Let  $D = \alpha(D_0)$  and  $D_0 \in \Pi_A^0$  with  $\alpha$  as in definition 3.21. Hence  $D_0 \xrightarrow[\text{IPC}]{\theta_0} D_0^{\dagger} \in \mathbf{N}(\vec{p}, \vec{q})$ . Then if we let  $\theta := \alpha \circ \theta_0$ , we have  $D \xrightarrow[\text{IPC}]{\theta} D^{\dagger} \in \mathbf{N}(\square)$  and thus  $D \in \downarrow \mathbf{N}(\square)$ .

2. P2 in definition 3.21 implies  $\llbracket \text{IPC, par} \rrbracket \vdash A_0 \triangleright A_0^{\text{p}}$  and lemma 3.20 implies  $\llbracket \text{IPC, parb} \rrbracket \vdash A \triangleright A_1^{\text{p}}$ .  $\square$

**Lemma 3.23.** *Given  $A \in \mathcal{L}_{\square}$ , for every  $\square E \in \text{sub}_o(A_1^{\text{p}})$  we have  $\square E \in \text{sub}_o(A)$ .*

*Proof.* Item P5 in definition 3.21.  $\square$

**Lemma 3.24.** *Given  $A \in \mathcal{L}_{\square}$ , for every  $D \in \Pi_A^1$  we have  $c_{\rightarrow}(D) \leq c_{\rightarrow}^{\square}(A)$ . (see section 2.5)*

*Proof.* If  $D = \alpha(D_0)$  with  $D_0 \in \Pi_A^0$ , by P4 in definition 3.21 we have  $c_{\rightarrow}(D_0) \leq c_{\rightarrow}^{\square}(A)$ . Also we have  $c_{\rightarrow}(D_0) = c_{\rightarrow}(\alpha(D_0))$  and thus  $c_{\rightarrow}(D) \leq c_{\rightarrow}^{\square}(A)$ .  $\square$

**Theorem 3.25.** (TYPE- $\alpha$ )  $(\downarrow \mathbf{N}(\square)^{\vee}, \top)$  is recursively strong downward compact and  $A_1^{\text{p}} = \lfloor A \rfloor_{\downarrow \mathbf{N}(\square)^{\vee}}^{\top}$ .

*Proof.* Recursively strong downward compactness is derived from definition of  $A_1^{\text{p}}$  once we show  $\lfloor A \rfloor_{\downarrow \mathbf{N}(\square)^{\vee}}^{\top} = A_1^{\text{p}}$ . Hence by theorem 2.16 it is enough to show the following items:

- $A_1^{\text{p}} \in \downarrow \mathbf{N}(\square)^{\vee}$ : First item of lemma 3.22.
- $\top \vdash A_1^{\text{p}} \rightarrow A$ : By P1 in definition 3.21 for every  $D_0 \in \Pi_A^0$  we have  $\text{IPC} \vdash D_0 \rightarrow A_0$ . Hence  $\text{IPC} \vdash D \rightarrow A$  for every  $D \in \alpha(\Pi_A^0)$ .
- $A \stackrel{\top}{\underset{\downarrow \mathbf{N}(\square)^{\vee}}{\approx}} A_1^{\text{p}}$ : By the second item of lemma 3.22 and remark 2.11 we have  $\llbracket \top, \text{parb} \rrbracket \vdash A \triangleright A_1^{\text{p}}$ . Then theorem 3.28 implies  $A \stackrel{\top}{\underset{\downarrow \mathbf{N}(\square)}{\approx}} A_1^{\text{p}}$ . Thus by theorem 2.15 we are done.  $\square$

**Corollary 3.26.** (TYPE- $\alpha$ )  $\top$  is closed under  $(\cdot)_1^{\text{p}}$ , i.e. if  $\top \vdash A$  then  $\top \vdash A_1^{\text{p}}$ .

*Proof.* Let  $\top \vdash A$  and hence  $\top \vdash \top \rightarrow A$ . Theorem 3.25 implies  $A \stackrel{\top}{\underset{\downarrow \mathbf{N}(\square)}{\approx}} A_1^{\text{p}}$ , and since  $\top \in \downarrow \mathbf{N}(\square)$ , we have  $\top \vdash \top \rightarrow A_1^{\text{p}}$ .  $\square$

**Theorem 3.27.** (TYPE- $\alpha$ ) For every  $A, B \in \mathcal{L}_{\square}$ :

$$\llbracket \top, \text{parb} \rrbracket \vdash A \triangleright B \quad \text{iff} \quad A \stackrel{\top}{\underset{\downarrow \mathbf{N}(\square)}{\approx}} B \quad \text{iff} \quad A \stackrel{\top}{\underset{\downarrow \mathbf{N}(\square)}{\approx}} B.$$



*Proof.*  $\llbracket \mathbb{T}, \text{parb} \rrbracket \vdash A \triangleright B$  implies  $A \Vdash_{\mathbb{N}(\square)}^{\mathbb{T}} B$ : theorem 3.28.

$A \Vdash_{\mathbb{N}(\square)}^{\mathbb{T}} B$  implies  $A \widetilde{\Vdash}_{\downarrow \mathbb{N}(\square)}^{\mathbb{T}} B$ : since  $\mathbb{T}$  is closed under outer substitutions, lemma 2.10 implies desired result.

$A \widetilde{\Vdash}_{\downarrow \mathbb{N}(\square)}^{\mathbb{T}} B$  implies  $\llbracket \mathbb{T}, \text{parb} \rrbracket \vdash A \triangleright B$ : let  $A \widetilde{\Vdash}_{\downarrow \mathbb{N}(\square)}^{\mathbb{T}} B$ . Corollary 2.17 and theorem 3.25 implies  $\mathbb{T} \vdash A_1^{\text{P}} \rightarrow B$  and hence  $\llbracket \mathbb{T}, \text{parb} \rrbracket \vdash A_1^{\text{P}} \triangleright B$ . On the other hand, lemma 3.22 implies  $\llbracket \mathbb{T}, \text{parb} \rrbracket \vdash A \triangleright A_1^{\text{P}}$ . Thus  $\llbracket \mathbb{T}, \text{parb} \rrbracket \vdash A \triangleright B$ , as desired.  $\square$

**Theorem 3.28 (Soundness).** (TYPE- $\alpha$ )  $\llbracket \mathbb{T}, \text{parb} \rrbracket \vdash A \triangleright B$  implies  $A \Vdash_{\mathbb{N}(\square)}^{\mathbb{T}} B$ .

*Proof.* Let  $\llbracket \mathbb{T}, \text{parb} \rrbracket \vdash A \triangleright B$ . Theorem 2.13 implies  $A \Vdash_{\text{PN}(\square)}^{\mathbb{T}} B$  and then corollary 3.3 implies the  $A \Vdash_{\mathbb{N}(\square)}^{\mathbb{T}} B$ .  $\square$

**Lemma 3.29.** (TYPE-0) Up to  $\mathbb{T}$ -provable equivalence we have  $\downarrow \mathbb{N}(\square)^{\vee} = \downarrow \text{PN}(\square)^{\vee}$ .

*Proof.* Since  $\downarrow \text{PN}(\square) \subseteq \downarrow \mathbb{N}(\square)$ , we have  $\downarrow \text{PN}(\square)^{\vee} \subseteq \downarrow \mathbb{N}(\square)^{\vee}$ . For other direction, let  $B \in \downarrow \mathbb{N}(\square)$ . Hence there is some substitution  $\theta$  such that  $B \xrightarrow{\theta}_{\mathbb{T}} B^{\dagger} \in \mathbb{N}(\square)$  and by theorem 3.2 there are propositions  $B_i^{\dagger} \in \text{PN}(\square)$  such that  $\text{iK4} \vdash B^{\dagger} \leftrightarrow \bigvee_i B_i^{\dagger}$ . Then let  $B_i := B \wedge B_i^{\dagger}$ . Since  $\mathbb{T} \vdash B \rightarrow B^{\dagger}$ , we have  $\mathbb{T} \vdash B \leftrightarrow \bigvee_i B_i$ . Then one may easily observe that  $B_i \xrightarrow{\theta}_{\mathbb{T}} B_i^{\dagger} \in \text{PN}(\square)$  and hence  $B_i \in \downarrow \text{PN}(\square)$ , as desired.  $\square$

In the following corollary, we consider  $\llbracket \mathbb{T}, \text{parb} \rrbracket$  as binary relation which it axiomatizes.

**Corollary 3.30.** (TYPE- $\alpha$ ) The following equalities hold. Moreover if  $\mathbb{T}$  is decidable, all above relations are decidable.

$$\llbracket \mathbb{T}, \text{parb} \rrbracket = \widetilde{\Vdash}_{\downarrow \mathbb{N}(\square)}^{\mathbb{T}} = \widetilde{\Vdash}_{\downarrow \mathbb{N}(\square)^{\vee}}^{\mathbb{T}} = \widetilde{\Vdash}_{\downarrow \text{PN}(\square)}^{\mathbb{T}} = \widetilde{\Vdash}_{\downarrow \text{PN}(\square)^{\vee}}^{\mathbb{T}} = \Vdash_{\mathbb{N}(\square)}^{\mathbb{T}} = \Vdash_{\mathbb{N}(\square)^{\vee}}^{\mathbb{T}} = \Vdash_{\text{PN}(\square)}^{\mathbb{T}} = \Vdash_{\text{PN}(\square)^{\vee}}^{\mathbb{T}}$$

*Proof.* Theorem 3.27 implies  $\llbracket \mathbb{T}, \text{parb} \rrbracket = \widetilde{\Vdash}_{\downarrow \mathbb{N}(\square)}^{\mathbb{T}}$ . On the other hand theorem 2.15 implies  $\widetilde{\Vdash}_{\downarrow \mathbb{N}(\square)}^{\mathbb{T}} = \widetilde{\Vdash}_{\downarrow \mathbb{N}(\square)^{\vee}}^{\mathbb{T}}$  and  $\widetilde{\Vdash}_{\downarrow \text{PN}(\square)}^{\mathbb{T}} = \widetilde{\Vdash}_{\downarrow \text{PN}(\square)^{\vee}}^{\mathbb{T}}$  and  $\Vdash_{\mathbb{N}(\square)}^{\mathbb{T}} = \Vdash_{\mathbb{N}(\square)^{\vee}}^{\mathbb{T}}$  and  $\Vdash_{\text{PN}(\square)}^{\mathbb{T}} = \Vdash_{\text{PN}(\square)^{\vee}}^{\mathbb{T}}$ . Lemma 3.29 implies  $\widetilde{\Vdash}_{\downarrow \mathbb{N}(\square)}^{\mathbb{T}} = \widetilde{\Vdash}_{\downarrow \text{PN}(\square)}^{\mathbb{T}}$  and also theorem 3.2 implies  $\Vdash_{\mathbb{N}(\square)}^{\mathbb{T}} = \Vdash_{\text{PN}(\square)^{\vee}}^{\mathbb{T}}$ .

Next we show decidability of  $A \widetilde{\Vdash}_{\downarrow \mathbb{N}(\square)^{\vee}}^{\mathbb{T}} B$ . Theorem 3.25 implies that  $[A]_{\downarrow \mathbb{N}(\square)^{\vee}}^{\mathbb{T}}$  exists and is computable. Then by corollary 2.17 it is enough to decide  $\mathbb{T} \vdash [A]_{\downarrow \mathbb{N}(\square)^{\vee}}^{\mathbb{T}} \rightarrow B$ , which is provided by decidability of  $\mathbb{T}$ .  $\square$

### 3.5 $\downarrow \text{SN}(\square)$ -Preservativity and $\text{SN}(\square)$ -Admissibility

In this section we show that  $\llbracket \mathbb{T}, \text{parb} \rrbracket \text{Le}^-$  axiomatizes  $\widetilde{\Vdash}_{\downarrow \text{SN}(\square)}^{\mathbb{T}}$  and  $\Vdash_{\text{SN}(\square)}^{\mathbb{T}}$  whenever  $\mathbb{T}$  is TYPE- $\alpha$ . Moreover we show that  $(\downarrow \text{SN}(\square)^{\vee}, \mathbb{T})$  is recursively strong downward compact whenever  $\mathbb{T}$  is TYPE- $\alpha$ .

**Lemma 3.31.** For every  $A \in \text{NNIL}(\square)$ , we have  $\llbracket \text{iK4}, \text{parb} \rrbracket \text{Le}^- \vdash A \triangleright A^{\square}$ .

*Proof.* Use induction on the complexity of  $A$ . All cases are similar to the proof of lemma 3.16 and left to the reader.  $\square$

**Definition 3.32.** For every  $A \in \mathcal{L}_{\square}$  define  $A^{\text{a}} \in \mathcal{L}_{\square}$  and  $\Pi'_A \subseteq \mathcal{L}_{\square}$  as follows.

$$\Pi'_A := \{D \wedge (D^{\dagger})^{\square} : D \in \Pi_A^1\} \quad \text{and} \quad A^{\text{a}} := \bigvee \Pi'_A.$$

Note that by definition 3.21, one may effectively compute  $\Pi'_A$  and  $A^{\text{a}}$ .

**Lemma 3.33.** If  $\mathbb{T} \supseteq \text{iK4}$  and  $B \in \downarrow \mathbb{N}(\square)$  then  $B \wedge (B^{\dagger})^{\square} \in \downarrow \text{SN}(\square)$  and  $\llbracket \mathbb{T}, \text{parb} \rrbracket \text{Le}^- \vdash B \triangleright B \wedge (B^{\dagger})^{\square}$ .

*Proof.* First observe that  $B \xrightarrow{\theta} B^\dagger$  implies  $B \wedge (B^\dagger)^\square \xrightarrow{\theta} (B^\dagger)^\square$ . Hence if  $B^\dagger \in \mathbf{N}(\square)$  then  $(B^\dagger)^\square \in \mathbf{SN}(\square)$ . Then remark 3.10 implies  $\mathbb{T} \vdash B \rightarrow B^\dagger$  and hence  $\llbracket \mathbb{T}, \text{parb} \rrbracket \vdash B \triangleright B^\dagger$ . Thus by lemma 3.31 and remark 2.11 we have  $\llbracket \mathbb{T}, \text{parb} \rrbracket \text{Le}^- \vdash B \triangleright (B^\dagger)^\square$ .  $\square$

**Lemma 3.34.** *Let  $\mathbb{T} \supseteq \text{iK4}$  and  $A \in \mathcal{L}_\square$ . Then*

1.  $\Pi'_A \subseteq \downarrow \mathbf{SN}(\square)$  and hence  $A^a \in \downarrow \mathbf{SN}(\square)^\vee$ .
2.  $\llbracket \mathbb{T}, \text{parb} \rrbracket \text{Le}^- \vdash A \triangleright A^a$ .

*Proof.* Lemmas 3.22 and 3.33.  $\square$

**Lemma 3.35.** *Let  $\mathbb{T} \supseteq \text{iK4}$  and  $B$  is a glb for  $A$  with respect to  $(\downarrow \mathbf{N}(\square)^\vee, \mathbb{T})$ . Also assume that  $B = \bigvee \Pi$  with  $\Pi \subseteq \downarrow \mathbf{N}(\square)$ . Then  $\bigvee \check{\Pi}$  is a glb for  $A$  with respect to  $(\downarrow \mathbf{SN}(\square)^\vee, \mathbb{T})$  in which  $\check{\Pi} := \{D \wedge (D^\dagger)^\square : D \in \Pi\}$ .*

*Proof.* By theorem 2.16 it is enough to show the following items for arbitrary  $D \in \Pi$ :

- $D \wedge (D^\dagger)^\square \in \downarrow \mathbf{SN}(\square)$ : Lemma 3.33.
- $\mathbb{T} \vdash [D \wedge (D^\dagger)^\square] \rightarrow A$ : By assumption we have  $\mathbb{T} \vdash D \rightarrow A$ . Then since  $\vdash [D \wedge (D^\dagger)^\square] \rightarrow D$  we get  $\mathbb{T} \vdash [D \wedge (D^\dagger)^\square] \rightarrow A$ .
- $A \underset{\downarrow \mathbf{SN}(\square)^\vee}{\overset{\mathbb{T}}{\approx}} \bigvee \check{\Pi}$ : By assumption we have  $A \underset{\downarrow \mathbf{N}(\square)^\vee}{\overset{\mathbb{T}}{\approx}} \bigvee \Pi$ . Also lemma 3.33 implies  $\llbracket \mathbb{T}, \text{parb} \rrbracket \text{Le}^- \vdash \bigvee \Pi \triangleright \bigvee \check{\Pi}$ . Thus theorem 3.39 implies  $\bigvee \Pi \underset{\downarrow \mathbf{SN}(\square)}{\overset{\mathbb{T}}{\approx}} \bigvee \check{\Pi}$  and hence  $A \underset{\downarrow \mathbf{SN}(\square)}{\overset{\mathbb{T}}{\approx}} \bigvee \check{\Pi}$ . Then by theorem 2.15 we have  $A \underset{\downarrow \mathbf{SN}(\square)^\vee}{\overset{\mathbb{T}}{\approx}} \bigvee \check{\Pi}$ .  $\square$

**Theorem 3.36.** (TYPE- $\alpha$ )  $(\downarrow \mathbf{SN}(\square)^\vee, \mathbb{T})$  is recursively strong downward compact and  $A^a = \lfloor A \rfloor_{\downarrow \mathbf{SN}(\square)^\vee}^\mathbb{T}$ .

*Proof.* Theorem 3.25 and lemma 3.35.  $\square$

**Corollary 3.37.** (TYPE- $\alpha$ )  $\mathbb{T}$  is closed under  $(\cdot)^a$ , i.e. if  $\mathbb{T} \vdash A$  then  $\mathbb{T} \vdash A^a$ .

*Proof.* Let  $\mathbb{T} \vdash A$  and hence  $\mathbb{T} \vdash \mathbb{T} \rightarrow A$ . Theorem 3.36 implies  $A \underset{\downarrow \mathbf{SN}(\square)}{\overset{\mathbb{T}}{\approx}} A^a$ , and since  $\mathbb{T} \in \downarrow \mathbf{SN}(\square)$ , we have  $\mathbb{T} \vdash \mathbb{T} \rightarrow A^a$ .  $\square$

**Theorem 3.38.** (TYPE- $\alpha$ ) For every  $A, B \in \mathcal{L}_\square$ :

$$\llbracket \mathbb{T}, \text{parb} \rrbracket \text{Le}^- \vdash A \triangleright B \quad \text{iff} \quad A \underset{\downarrow \mathbf{SN}(\square)}{\overset{\mathbb{T}}{\approx}} B \quad \text{iff} \quad A \underset{\downarrow \mathbf{SN}(\square)}{\overset{\mathbb{T}}{\approx}} B.$$

*Proof.*  $\llbracket \mathbb{T}, \text{parb} \rrbracket \text{Le}^- \vdash A \triangleright B$  implies  $A \underset{\downarrow \mathbf{SN}(\square)}{\overset{\mathbb{T}}{\approx}} B$ : theorem 3.39.

$A \underset{\downarrow \mathbf{SN}(\square)}{\overset{\mathbb{T}}{\approx}} B$  implies  $A \underset{\downarrow \mathbf{SN}(\square)}{\overset{\mathbb{T}}{\approx}} B$ : since  $\mathbb{T}$  is closed under outer substitutions, lemma 2.10 implies desired result.

$A \underset{\downarrow \mathbf{SN}(\square)}{\overset{\mathbb{T}}{\approx}} B$  implies  $\llbracket \mathbb{T}, \text{parb} \rrbracket \text{Le}^- \vdash A \triangleright B$ : let  $A \underset{\downarrow \mathbf{SN}(\square)}{\overset{\mathbb{T}}{\approx}} B$ . Corollary 2.17 and theorem 3.36 implies  $\mathbb{T} \vdash A^a \rightarrow B$  and hence  $\llbracket \mathbb{T}, \text{parb} \rrbracket \text{Le}^- \vdash A^a \triangleright B$ . On the other hand, lemma 3.34 implies  $\llbracket \mathbb{T}, \text{parb} \rrbracket \text{Le}^- \vdash A \triangleright A^a$ . Thus  $\llbracket \mathbb{T}, \text{parb} \rrbracket \text{Le}^- \vdash A \triangleright B$ .  $\square$

**Theorem 3.39 (Soundness).** (TYPE- $\alpha$ )  $\llbracket \mathbb{T}, \text{parb} \rrbracket \text{Le}^- \vdash A \triangleright B$  implies  $A \underset{\downarrow \mathbf{SN}(\square)}{\overset{\mathbb{T}}{\approx}} B$ .

*Proof.* Let  $\llbracket \mathbb{T}, \text{parb} \rrbracket \text{Le}^- \vdash A \triangleright B$ . Theorem 2.13 implies  $A \underset{\downarrow \mathbf{SPN}(\square)}{\overset{\mathbb{T}}{\approx}} B$  and corollary 3.3 implies  $A \underset{\downarrow \mathbf{SN}(\square)}{\overset{\mathbb{T}}{\approx}} B$ .  $\square$

**Lemma 3.40.** (TYPE- $\alpha$ ) Up to  $\mathbb{T}$ -provable equivalence we have  $\downarrow \mathbf{SN}(\square)^\vee = \downarrow \mathbf{SPN}(\square)^\vee$  and  $\mathbf{C}\downarrow \mathbf{SN}(\square)^\vee = \mathbf{C}\downarrow \mathbf{SPN}(\square)^\vee$ .

*Proof.* We only reason for the first equality and leave similar argument for second one to the reader. Since  $\downarrow\text{SPN}(\square) \subseteq \downarrow\text{SN}(\square)$ , we have  $\downarrow\text{SPN}(\square)^\vee \subseteq \downarrow\text{SN}(\square)^\vee$ . For other direction, let  $B \in \downarrow\text{SN}(\square)$ . Hence there is some substitution  $\theta$  such that  $B \xrightarrow{\theta} B^\dagger \in \text{SN}(\square)$  and by theorem 3.2 there are propositions  $B_i^\dagger \in \text{SPN}(\square)$  such that  $\text{iK4} \vdash B^\dagger \leftrightarrow \bigvee_i B_i^\dagger$ . Then let  $B_i := B \wedge B_i^\dagger$ . Since  $\top \vdash B \rightarrow B^\dagger$ , we have  $\top \vdash B \leftrightarrow \bigvee_i B_i$ . Then one may easily observe that  $B_i \xrightarrow{\theta} B_i^\dagger \in \text{SPN}(\square)$  and hence  $B_i \in \downarrow\text{SPN}(\square)$ .  $\square$

In the following corollary, we consider  $\llbracket \top, \text{parb} \rrbracket \text{Le}^-$  as binary relation which it axiomatizes.

**Corollary 3.41.** (TYPE- $\alpha$ ) *The following equalities hold. Moreover if  $\top$  is decidable, all above relations are decidable.*

$$\llbracket \top, \text{parb} \rrbracket \text{Le}^- = \overset{\top}{\underset{\downarrow\text{SN}(\square)}{\approx}} = \overset{\top}{\underset{\downarrow\text{SN}(\square)^\vee}{\approx}} = \overset{\top}{\underset{\downarrow\text{SPN}(\square)}{\approx}} = \overset{\top}{\underset{\downarrow\text{SPN}(\square)^\vee}{\approx}} = \overset{\top}{\underset{\text{SN}(\square)}{\approx}} = \overset{\top}{\underset{\text{SN}(\square)^\vee}{\approx}} = \overset{\top}{\underset{\text{SPN}(\square)}{\approx}} = \overset{\top}{\underset{\text{SPN}(\square)^\vee}{\approx}}$$

*Proof.* Theorem 3.38 implies  $\llbracket \top, \text{parb} \rrbracket \text{Le}^- = \overset{\top}{\underset{\downarrow\text{SN}(\square)}{\approx}} = \overset{\top}{\underset{\text{SN}(\square)}{\approx}}$ . On the other hand theorem 2.15 implies  $\overset{\top}{\underset{\downarrow\text{SN}(\square)}{\approx}} = \overset{\top}{\underset{\downarrow\text{SN}(\square)^\vee}{\approx}}$  and  $\overset{\top}{\underset{\downarrow\text{SPN}(\square)}{\approx}} = \overset{\top}{\underset{\downarrow\text{SPN}(\square)^\vee}{\approx}}$  and  $\overset{\top}{\underset{\text{SN}(\square)}{\approx}} = \overset{\top}{\underset{\text{SN}(\square)^\vee}{\approx}}$  and  $\overset{\top}{\underset{\text{SPN}(\square)}{\approx}} = \overset{\top}{\underset{\text{SPN}(\square)^\vee}{\approx}}$ . Moreover theorem 3.2 implies  $\overset{\top}{\underset{\text{SN}(\square)}{\approx}} = \overset{\top}{\underset{\text{SPN}(\square)^\vee}{\approx}}$ . Finally lemma 3.40 implies  $\overset{\top}{\underset{\downarrow\text{SN}(\square)^\vee}{\approx}} = \overset{\top}{\underset{\downarrow\text{SPN}(\square)^\vee}{\approx}}$ .

Next we show decidability of  $A \overset{\top}{\underset{\downarrow\text{SN}(\square)^\vee}{\approx}} B$ . Theorem 3.36 implies that  $\llbracket A \rrbracket_{\downarrow\text{SN}(\square)^\vee}^\top$  exists and is computable. Then by corollary 2.17 it is enough to decide  $\top \vdash \llbracket A \rrbracket_{\downarrow\text{SN}(\square)^\vee}^\top \rightarrow B$ , which is provided by decidability of  $\top$ .  $\square$

### 3.6 $\text{C}\downarrow\text{SN}(\square)$ -Preservativity

In this section we show that  $\llbracket \top, \text{parb} \rrbracket \text{Le}$  axiomatizes  $\overset{\top}{\underset{\text{C}\downarrow\text{SN}(\square)}{\approx}}$  whenever  $\top$  is TYPE- $\alpha$ . Moreover we show that  $(\text{C}\downarrow\text{SN}(\square)^\vee, \top)$  is recursively strong downward compact whenever  $\top$  is TYPE- $\alpha$ .

**Definition 3.42.** For every  $A \in \mathcal{L}_\square$  we define  $A^p \in \mathcal{L}_\square$  and  $\Pi_A \subseteq \mathcal{L}_\square$  as follows.

$$\Pi_A := \{ \square D \wedge (D^\dagger)^\square : D \in \Pi_A^1 \} \quad \text{and} \quad A^p := \bigvee \Pi_A.$$

Note that by definition 3.21, one may effectively compute  $\Pi_A$  and  $A^p$ .

**Lemma 3.43.** *If  $\top \supseteq \text{iK4}$  and  $B \in \downarrow\text{N}(\square)$  then  $\square B \wedge (B^\dagger)^\square \in \text{C}\downarrow\text{SN}(\square)$  and  $\llbracket \top, \text{parb} \rrbracket \text{Le} \vdash B \triangleright \square B \wedge (B^\dagger)^\square$ .*

*Proof.* First observe that  $B \xrightarrow{\theta} B^\dagger \in \text{N}(\square)$  implies  $\square B \wedge (B^\dagger)^\square \xrightarrow{\theta} \square B \wedge (B^\dagger)^\square$ . Hence if  $B^\dagger \in \text{N}(\square)$  then  $\square B \wedge (B^\dagger)^\square \in \text{SN}(\square)$  and thus  $\square B \wedge (B^\dagger)^\square \in \text{C}\downarrow\text{SN}(\square)$ . Lemma 3.33 implies  $\llbracket \top, \text{parb} \rrbracket \text{Le} \vdash B \triangleright (B^\dagger)^\square$ . Also by Leivant's principle we have  $\llbracket \top, \text{parb} \rrbracket \text{Le} \vdash B \triangleright \square B$  and thus  $\llbracket \top, \text{parb} \rrbracket \text{Le} \vdash B \triangleright \square B \wedge (B^\dagger)^\square$ .  $\square$

**Lemma 3.44.** *Let  $\top \supseteq \text{iK4}$  and  $A \in \mathcal{L}_\square$ . Then we have:*

1.  $\Pi_A \subseteq \text{C}\downarrow\text{SN}(\square)$  and hence  $A^p \in \text{C}\downarrow\text{SN}(\square)^\vee$ .
2.  $\llbracket \top, \text{parb} \rrbracket \text{Le} \vdash A \triangleright A^p$ .

*Proof.* Lemmas 3.22 and 3.43.  $\square$

**Lemma 3.45.** *Let  $\top \supseteq \text{iK4}$  and  $B$  is a glb for  $A$  with respect to  $(\downarrow\text{N}(\square)^\vee, \top)$ . Also assume that  $B = \bigvee \Pi$  with  $\Pi \subseteq \downarrow\text{N}(\square)$ . Then  $\bigvee \hat{\Pi}$  is glb for  $A$  with respect to  $(\text{C}\downarrow\text{SN}(\square)^\vee, \top)$ , in which  $\hat{\Pi} := \{ \square D \wedge (D^\dagger)^\square : D \in \Pi \}$ .*

*Proof.* By theorem 2.16 it is enough to show the following items for arbitrary  $D \in \Pi$ :

- $\Box D \wedge (D^\dagger)^\Box \in \mathsf{C}\downarrow\mathsf{SN}(\Box)$ : Lemma 3.43.
- $\top \vdash [\Box D \wedge (D^\dagger)^\Box] \rightarrow A$ : By assumption we have  $\top \vdash D \rightarrow A$ . Then since  $\vdash [\Box D \wedge (D^\dagger)^\Box] \rightarrow D$  we get  $\top \vdash [\Box D \wedge (D^\dagger)^\Box] \rightarrow A$ .
- $A \overset{\top}{\underset{\mathsf{C}\downarrow\mathsf{SN}(\Box)^\vee}{\rightsquigarrow}} \bigvee \hat{\Pi}$ : By assumption we have  $A \overset{\top}{\underset{\mathsf{C}\downarrow\mathsf{SN}(\Box)^\vee}{\rightsquigarrow}} \bigvee \hat{\Pi}$ . Also lemma 3.43 implies  $\llbracket \top, \text{parb} \rrbracket \text{Le} \vdash \bigvee \hat{\Pi} \triangleright \bigvee \hat{\Pi}$ . Thus theorem 3.49 implies  $\bigvee \hat{\Pi} \overset{\top}{\underset{\mathsf{C}\downarrow\mathsf{SN}(\Box)}{\rightsquigarrow}} \bigvee \hat{\Pi}$  and hence  $A \overset{\top}{\underset{\mathsf{C}\downarrow\mathsf{SN}(\Box)}{\rightsquigarrow}} \bigvee \hat{\Pi}$ . Then by theorem 2.15 we have  $A \overset{\top}{\underset{\mathsf{C}\downarrow\mathsf{SN}(\Box)^\vee}{\rightsquigarrow}} \bigvee \hat{\Pi}$ .  $\square$

**Theorem 3.46.**  $(\text{TYPE-}\alpha)(\mathsf{C}\downarrow\mathsf{SN}(\Box)^\vee, \top)$  is recursively strong downward compact and  $A^p = \lfloor A \rfloor_{\mathsf{C}\downarrow\mathsf{SN}(\Box)^\vee}^\top$ .

*Proof.* Theorem 3.25 and lemma 3.45.  $\square$

**Corollary 3.47.**  $(\text{TYPE-}\alpha)\top$  is closed under  $(\cdot)^p$ , i.e. if  $\top \vdash A$  then  $\top \vdash A^p$ .

*Proof.* Let  $\top \vdash A$  and hence  $\top \vdash \top \rightarrow A$ . Theorem 3.46 implies  $A \overset{\top}{\underset{\mathsf{C}\downarrow\mathsf{SN}(\Box)}{\rightsquigarrow}} A^p$ , and since  $\top \in \mathsf{C}\downarrow\mathsf{SN}(\Box)$ , we have  $\top \vdash \top \rightarrow A^p$ .  $\square$

**Theorem 3.48.**  $(\text{TYPE-}\alpha)$  For every  $A, B \in \mathcal{L}_\Box$ :

$$\llbracket \top, \text{parb} \rrbracket \text{Le} \vdash A \triangleright B \quad \text{iff} \quad A \overset{\top}{\underset{\mathsf{C}\downarrow\mathsf{SN}(\Box)}{\rightsquigarrow}} B.$$

*Proof.*  $\llbracket \top, \text{parb} \rrbracket \text{Le} \vdash A \triangleright B$  implies  $A \overset{\top}{\underset{\mathsf{C}\downarrow\mathsf{SN}(\Box)}{\rightsquigarrow}} B$ : theorem 3.49.

$A \overset{\top}{\underset{\mathsf{C}\downarrow\mathsf{SN}(\Box)}{\rightsquigarrow}} B$  implies  $\llbracket \top, \text{parb} \rrbracket \text{Le} \vdash A \triangleright B$ : let  $A \overset{\top}{\underset{\mathsf{C}\downarrow\mathsf{SN}(\Box)}{\rightsquigarrow}} B$ . Corollary 2.17 and theorem 3.46 imply  $\top \vdash A^p \rightarrow B$  and hence  $\llbracket \top, \text{parb} \rrbracket \text{Le} \vdash A^p \triangleright B$ . On the other hand, lemma 3.44 implies  $\llbracket \top, \text{parb} \rrbracket \text{Le} \vdash A \triangleright A^p$ . Thus  $\llbracket \top, \text{parb} \rrbracket \text{Le} \vdash A \triangleright B$ .  $\square$

**Theorem 3.49 (Soundness).**  $(\text{TYPE-}\alpha)\llbracket \top, \text{parb} \rrbracket \text{Le} \vdash A \triangleright B$  implies  $A \overset{\top}{\underset{\mathsf{C}\downarrow\mathsf{SN}(\Box)}{\rightsquigarrow}} B$ .

*Proof.* Let  $\llbracket \top, \text{parb} \rrbracket \text{Le} \vdash A \triangleright B$ . Theorem 2.12 implies  $A \overset{\top}{\underset{\mathsf{C}\downarrow\mathsf{SPN}(\Box)}{\rightsquigarrow}} B$  and thus lemma 3.40 implies  $A \overset{\top}{\underset{\mathsf{C}\downarrow\mathsf{SN}(\Box)}{\rightsquigarrow}} B$ .  $\square$

In the following corollary, we consider  $\llbracket \top, \text{parb} \rrbracket \text{Le}$  as binary relation which it axiomatizes.

**Corollary 3.50.**  $(\text{TYPE-}\alpha)\llbracket \top, \text{parb} \rrbracket \text{Le} = \overset{\top}{\underset{\mathsf{C}\downarrow\mathsf{SN}(\Box)}{\rightsquigarrow}} = \overset{\top}{\underset{\mathsf{C}\downarrow\mathsf{SN}(\Box)^\vee}{\rightsquigarrow}} = \overset{\top}{\underset{\mathsf{C}\downarrow\mathsf{SPN}(\Box)}{\rightsquigarrow}} = \overset{\top}{\underset{\mathsf{C}\downarrow\mathsf{SPN}(\Box)^\vee}{\rightsquigarrow}}$ . Moreover if  $\top$  is decidable, all mentioned relations are decidable.

*Proof.* Theorem 3.48 implies  $\llbracket \top, \text{parb} \rrbracket \text{Le} = \overset{\top}{\underset{\mathsf{C}\downarrow\mathsf{SN}(\Box)}{\rightsquigarrow}}$ . On the other hand theorem 2.15 implies  $\overset{\top}{\underset{\mathsf{C}\downarrow\mathsf{SN}(\Box)}{\rightsquigarrow}} = \overset{\top}{\underset{\mathsf{C}\downarrow\mathsf{SN}(\Box)^\vee}{\rightsquigarrow}}$  and  $\overset{\top}{\underset{\mathsf{C}\downarrow\mathsf{SN}(\Box)}{\rightsquigarrow}} = \overset{\top}{\underset{\mathsf{C}\downarrow\mathsf{SPN}(\Box)}{\rightsquigarrow}}$ . Finally lemma 3.40 implies  $\overset{\top}{\underset{\mathsf{C}\downarrow\mathsf{SN}(\Box)^\vee}{\rightsquigarrow}} = \overset{\top}{\underset{\mathsf{C}\downarrow\mathsf{SPN}(\Box)^\vee}{\rightsquigarrow}}$ .

Next we show decidability of  $A \overset{\top}{\underset{\mathsf{C}\downarrow\mathsf{SN}(\Box)^\vee}{\rightsquigarrow}} B$ . Theorem 3.46 implies that  $\lfloor A \rfloor_{\mathsf{C}\downarrow\mathsf{SN}(\Box)^\vee}^\top$  exists and is computable. Then by corollary 2.17 it is enough to decide  $\top \vdash \lfloor A \rfloor_{\mathsf{C}\downarrow\mathsf{SN}(\Box)^\vee}^\top \rightarrow B$ , which is provided by decidability of  $\top$ .  $\square$

**Lemma 3.51.**  $\text{iGLH} \vdash A$  implies  $\text{iPH} \vdash A$ .

*Proof.* We prove by induction on the proof complexity of  $\text{iGLH} \vdash A$  that  $\text{iPH}_\Box \vdash A$ . All cases are trivial except for when  $A$  is an axiom instance of  $\mathsf{H}(\mathsf{C}\downarrow\mathsf{SN}(\Box), \text{iGL})$ , i.e.  $A = \Box B \rightarrow \Box C$  with  $B \overset{\text{iGL}}{\underset{\mathsf{C}\downarrow\mathsf{SN}(\Box)}{\rightsquigarrow}} C$ . Corollary 3.50 implies that  $\llbracket \text{iGL}, \text{parb} \rrbracket \text{Le} \vdash B \triangleright C$ . Also from definition of  $\llbracket \text{iGL}, \text{parb} \rrbracket \text{Le}$  (see section 2.10) it is clear that  $\text{iPH} \vdash B \triangleright C$ . Then Cut implies  $\text{iPH} \vdash \top \triangleright B \rightarrow \top \triangleright C$ . Thus  $\text{iPH}_\Box \vdash \Box B \rightarrow \Box C$ .  $\square$

### 3.7 SN( $\square$ )-Preservativity

In this section we show that  $\llbracket \mathbb{T}, \text{parb} \rrbracket \text{LeA}$  axiomatizes  $\overset{\mathbb{T}}{\cong}_{\text{SN}(\square)}$  whenever  $\mathbb{T}$  is TYPE- $\alpha$ . Moreover we show that  $(\text{SN}(\square)^\vee, \mathbb{T})$  is recursively strong downward compact whenever  $\mathbb{T}$  is TYPE- $\alpha$ .

**Theorem 3.52.** *For every  $A \in \mathcal{L}_0$  one may effectively compute  $A^* \in \text{NNIL}(\text{par})$  such that:*

1.  $\text{IPC} \vdash A^* \rightarrow A$ ,
2.  $\llbracket \text{IPC}, \text{par} \rrbracket A \vdash A \triangleright A^*$ ,
3.  $\text{sub}^{\text{atom}}(A^*) \subseteq \text{sub}^{\text{atom}}(A)$ .

*Proof.* See lemma 4.23 in [Mojtahedi, 2022a]. □

Remember that  $\text{parb}$  is the set of parameters or boxed propositions.

**Lemma 3.53.** *Let  $\alpha$  be a substitution such that for every  $a \in \text{atom}$  if  $\alpha(a) \neq a$  then  $a \in \text{par}$  and  $\alpha(a) \in \text{parb}$ . Then  $\llbracket \text{IPC}, \text{par} \rrbracket A \vdash A \triangleright B$  implies  $\llbracket \text{IPC}, \text{parb} \rrbracket A \vdash \alpha(A) \triangleright \alpha(B)$ , for every  $A, B \in \mathcal{L}_0$ .*

*Proof.* Straightforward induction on the proof  $\llbracket \text{IPC}, \text{par} \rrbracket A \vdash A \triangleright B$  and left to reader. □

**Lemma 3.54.** *For every  $A \in \mathcal{L}_\square$  one may effectively compute some  $A^{**} \in \text{NNIL}(\square)$  such that:*

1.  $\text{IPC} \vdash A^{**} \rightarrow A$ ,
2.  $\llbracket \text{IPC}, \text{parb} \rrbracket \text{LeA} \vdash A \triangleright A^{**}$ ,
3.  $\text{sub}^{\text{atom}}(A^{**}) \subseteq \text{sub}(A)$ .

*Proof.* Let  $\square B_1, \dots, \square B_n$  be the list of all outer occurrences of boxed formulas in  $A$ . Moreover let  $\vec{p} := p_1, \dots, p_n$  be a list of fresh atomic parameters, i.e.  $p_i \notin \text{sub}(A)$  and they are pairwise distinct. Also assume that  $\alpha$  be the substitution such that

$$\alpha(a) := \begin{cases} \square B_i & : \text{ if } a = p_i, \\ a & : \text{ otherwise.} \end{cases}$$

Then there is a unique  $A_0 \in \mathcal{L}_0$  such that  $\alpha(A_0) = A$ . Theorem 3.52 give us some  $A_0^* \in \text{NNIL}(\text{par})$  such that

1.  $\text{IPC} \vdash A_0^* \rightarrow A_0$ ,
2.  $\llbracket \text{IPC}, \text{par} \rrbracket A \vdash A_0 \triangleright A_0^*$ ,
3.  $\text{sub}^{\text{atom}}(A_0^*) \subseteq \text{sub}^{\text{atom}}(A_0)$ .

Define  $A^{**} := \alpha(A_0^*)$ . Then we have:

1.  $\text{IPC} \vdash A^{**} \rightarrow A$ ,
2. By lemma 3.53 we have  $\llbracket \text{IPC}, \text{parb} \rrbracket \text{LeA} \vdash A \triangleright A^{**}$ ,
3.  $\text{sub}^{\text{atom}}(A^{**}) \subseteq \text{sub}(A)$ . □

**Theorem 3.55.** *For every  $A$  there is some  $A^s \in \text{SN}(\square)$  such that:*

1.  $\text{iK4} \vdash A^s \rightarrow A$ ,
2.  $\llbracket \text{iK4}, \text{parb} \rrbracket \text{LeA} \vdash A \triangleright A^s$ ,
3.  $\square B \in \text{sub}(A^s)$  implies either  $\square B \in \text{sub}(A)$  or  $\square B \in \text{SN}(\square)$ .

4.  $\text{sub}^{\text{atom}}(A^s) \subseteq \text{sub}^{\text{atom}}(A)$ .

*Proof.* By lemma 3.54 there is some  $A^{**}$  with mentioned properties. Since  $A^{**} \in \mathbf{N}(\square)$  we have  $\text{iK4} \vdash (A^{**})^\square \rightarrow A^{**}$ . Also lemma 3.31 implies  $\llbracket \text{iK4, parb} \rrbracket \text{Le} \vdash A^{**} \triangleright (A^{**})^\square$ . Hence if we let  $A^s := (A^{**})^\square$ , by lemma 3.54 we have all required properties.  $\square$

**Theorem 3.56.**  $(\text{TYPE-}\alpha)(\text{SN}(\square), \mathbb{T})$  is recursively strong downward compact and  $A^s := (A^{**})^\square = \lfloor A \rfloor_{\text{SN}(\square)}^\mathbb{T}$ .

*Proof.* We show that  $A^s$  is the  $(\text{SN}(\square), \mathbb{T})$ -glb for  $A$  in which  $A^s$  is as provided by theorem 3.55. By theorem 3.55,  $A^s \in \text{SN}(\square)$ ,  $\text{iK4} \vdash A^s \rightarrow A$  and  $\llbracket \mathbb{T, parb} \rrbracket \text{Le} \vdash A \triangleright A^s$ . Then theorem 3.58 implies  $A \stackrel{\mathbb{T}}{\underset{\text{SN}(\square)}{\rightsquigarrow}} A^s$ . Hence theorem 2.16 implies desired result.  $\square$

**Theorem 3.57.**  $(\text{TYPE-}\alpha)\llbracket \mathbb{T, parb} \rrbracket \text{Le} \text{A}$  is sound and complete for  $\text{SN}(\square)$ -preservativity in  $\mathbb{T}$ , i.e. for every  $A, B \in \mathcal{L}_\square$

$$\llbracket \mathbb{T, parb} \rrbracket \text{Le} \vdash A \triangleright B \quad \text{iff} \quad A \stackrel{\mathbb{T}}{\underset{\text{SN}(\square)}{\rightsquigarrow}} B.$$

*Proof.* The left-to-right direction (soundness) holds by theorem 3.58.

For the other direction (completeness), let  $A \stackrel{\mathbb{T}}{\underset{\text{SN}(\square)}{\rightsquigarrow}} B$ . Then theorem 3.56 and corollary 2.17 implies  $\mathbb{T} \vdash A^s \rightarrow B$  and hence  $\llbracket \mathbb{T, parb} \rrbracket \text{Le} \vdash A^s \triangleright B$ . Also by theorem 3.55 we have  $\llbracket \mathbb{T, parb} \rrbracket \text{Le} \vdash A \triangleright A^s$  and cut implies desired result.  $\square$

**Theorem 3.58 (Soundness).**  $(\text{TYPE-}\alpha)\llbracket \mathbb{T, parb} \rrbracket \text{Le} \vdash A \triangleright B$  implies  $A \stackrel{\mathbb{T}}{\underset{\text{SN}(\square)}{\rightsquigarrow}} B$ .

*Proof.* Let  $\llbracket \mathbb{T, parb} \rrbracket \text{Le} \vdash A \triangleright B$ . Theorem 2.12 implies  $A \stackrel{\mathbb{T}}{\underset{\text{SPN}(\square)}{\rightsquigarrow}} B$  and thus corollary 3.3 implies  $A \stackrel{\mathbb{T}}{\underset{\text{SN}(\square)}{\rightsquigarrow}} B$ .  $\square$

For the uniformity of notations in the following corollary we consider  $\llbracket \mathbb{T, parb} \rrbracket \text{Le} \text{A}$  as the binary relation which it axiomatizes.

**Corollary 3.59.**  $(\text{TYPE-}\alpha)\llbracket \mathbb{T, parb} \rrbracket \text{Le} \text{A} = \stackrel{\mathbb{T}}{\underset{\text{SN}(\square)}{\rightsquigarrow}} = \stackrel{\mathbb{T}}{\underset{\text{SPN}(\square)}{\rightsquigarrow}} = \stackrel{\mathbb{T}}{\underset{\text{SPN}(\square)^\vee}{\rightsquigarrow}}$ . Moreover If  $\mathbb{T}$  is decidable, then all mentioned relations are decidable.

*Proof.* All equalities are derived by theorems 2.15 and 3.57 and corollary 3.3. For the decidability of  $\stackrel{\mathbb{T}}{\underset{\text{SN}(\square)}{\rightsquigarrow}}$  we have following argument. Theorem 3.56 implies that  $\lfloor A \rfloor_{\text{SN}(\square)}^\mathbb{T}$  exists and is computable. Then by corollary 2.17 it is enough to decide  $\mathbb{T} \vdash \lfloor A \rfloor_{\text{SN}(\square)}^\mathbb{T} \rightarrow B$ , which is provided by decidability of  $\mathbb{T}$ .  $\square$

## 4 Provability and preservativity semantics

[Iemhoff, 2001c,a, 2003] consider some Kripke semantic for the preservativity logic  $\text{iPH}$  and proves its soundness/completeness theorems. Although such Kripke semantics are interesting tools, the main obstacle for their usage is that they are infinite. Here we introduce another semantics for provability and preservativity logics. This variant, as we will see later, enjoys the finite model property which is the key point for proving arithmetical completeness for the provability logic of  $\text{HA}$  and its decidability. The idea behind provability and preservativity semantics is that it recursively assigns a theory to every accessible node and defines validity of  $\square A$  in this way: for every accessible node we have the *provability* of  $A$  in the assigned theory. Since the assigned theory is defined via recurse on the accessibility relationship, we must require that the accessibility relation is conversely well-founded.

**Definition 4.1.** Let  $\mathbb{T}$  be a modal logic and  $\Delta \subseteq \Gamma \subseteq \mathcal{L}_\triangleright$  sets of propositions such that  $\Delta$  and  $\Gamma$  are closed under  $\Delta$ -conjunctions, i.e.  $A \in \Gamma$  ( $A \in \Delta$ ) and  $B \in \Delta$  implies  $A \wedge B \in \Gamma$  ( $A \wedge B \in \Delta$ ). A quintuple  $\mathcal{K} = (W, \preceq, \square, V, \varphi)$  is called a  $(\Delta, \Gamma, \mathbb{T})$ -semantic if  $\tilde{\mathcal{K}} := (W, \preceq, \square, V)$  is a transitive conversely well-founded Kripke model for the intuitionistic modal logic (as defined in section 2.6) and

- $\varphi$  is a function with the domain  $W\sqsubset$ , i.e. the set of all  $\sqsubset$ -accessible nodes in  $W$ .
- $\varphi(w) \in \Gamma$ . For the sake of simplicity of notations we may write  $\varphi_w$  for  $\varphi(w)$ .
- $\mathcal{K}$  satisfies compatibility, i.e. for every  $u \in W\sqsubset$  we have  $\mathcal{K}, u \Vdash \varphi_u$ , in which  $\Vdash$  is the forcing relation that will be defined below.

The forcing relation  $\mathcal{K}, w \Vdash A$  is defined by induction on  $A$  as we have in intuitionistic non-modal language, and when  $A = B \triangleright C$  is defined as follows:

$$\mathcal{K}, w \Vdash B \triangleright C \quad \Leftrightarrow \quad \forall u \sqsubset w \forall E \in \Delta \ (\Phi_u \vdash_{\top} E \rightarrow B \text{ implies } \Phi_u \vdash_{\top} E \rightarrow C),$$

in which  $\Phi_u := \{\varphi_u\} \cup \Delta_u^{\mathcal{K}}$  and  $\Delta_u^{\mathcal{K}}$  is defined in the following line. Given a set of modal propositions  $Y$ , define

$$Y_w^{\mathcal{K}} := \{E \in Y : \mathcal{K}, w \Vdash E\} \quad \text{and} \quad \bar{Y}_w^{\mathcal{K}} := \{E \in Y : \mathcal{K}, w \Vdash^+ E\}.$$

For the sake of simplicity, whenever no confusion is likely we may omit the superscript  $\mathcal{K}$  and simply write  $Y_w$  and  $\bar{Y}_w$  instead. Also the notation  $\mathcal{K}, w \Vdash^+ A$  is a shorthand for

“there is some  $u \sqsubset w$  such that  $\mathcal{K}, w' \Vdash A$  for every  $w' \sqsubset u$ .”

We also define the following properties for preservativity semantics:

- $\mathcal{K}$  is called  $Y$ -full if  $\Phi_u \vdash_{\top} Y_u$  for every  $u \in W\sqsubset$ .  $\mathcal{K}$  is called full if it is  $\Gamma$ -full. We say that  $\mathcal{K}$  is  $A$ -full if it is  $Y$ -full for  $Y := \{B, C : B \triangleright C \in \text{sub}(A)\}$ .
- $\mathcal{K}$  is called weakly  $Y$ -full if  $\Phi_u \vdash_{\top} \bar{Y}_u$  for every  $u \in W\sqsubset$ . Note that since  $\bar{Y}_u \subseteq Y_u$ , weakly  $Y$ -fullness is implied from  $Y$ -fullness.  $\mathcal{K}$  is called weakly full if it is weakly  $\Gamma$ -full. We say that  $\mathcal{K}$  is weakly  $A$ -full if it is weakly  $Y$ -full for  $Y := \{B : \Box B \in \text{sub}(A)\}$ .
- We say that  $\mathcal{K}$  has a property of intuitionistic modal Kripke models (like transitive) if  $\tilde{\mathcal{K}}$  is so.

Whenever  $\Gamma = \Delta$  we simply say that  $\mathcal{K}$  is a  $(\Gamma, \top)$ -semantic. Note that in this case we may assume that  $\varphi_w = \top$  for every  $w \in W\sqsubset$ . Whenever the triple  $(\Delta, \Gamma, \top)$  may be inferred from context or we do not care what they are, we may simply say that  $\mathcal{K}$  is a preservativity semantic.

**Remark 4.2.** Since non-modal connectives of propositional language are treated as usual in Kripke models of intuitionistic logic, for every preservativity model  $\mathcal{K}$  we have  $\mathcal{K} \Vdash \text{IPC}$ .

**Remark 4.3.** Whenever we restrict ourselves to the language  $\mathcal{L}_{\sqsubset}$ , definition simplifies to following (remember that  $\Box B = \top \triangleright B$ ):

$$\mathcal{K}, w \Vdash \Box B \quad \Leftrightarrow \quad \forall u \sqsubset w \ \Phi_u \vdash_{\top} B.$$

Whenever  $(\Gamma, \top)$  is downward compact, the preservativity semantics gets a more elegant form:

**Lemma 4.4.** Let  $(\Gamma, \top)$  be downward compact,  $\mathcal{K} := (W, \succ, \sqsubset, V, \varphi)$  an  $A$ -full  $(\Delta, \Gamma, \top)$ -semantic and  $\mathcal{K} \Vdash \top$ . Then for every  $w \in W$  and  $\Box C \in \text{sub}(A)$  we have

$$\mathcal{K}, w \Vdash \Box C \quad \Leftrightarrow \quad \forall u \sqsubset w \ \mathcal{K}, u \Vdash [C]_{\Gamma}^{\top}.$$

*Proof.* Left-to-right: Let  $\mathcal{K}, w \Vdash \Box C$  and  $u \sqsubset w$  seeking to show  $\mathcal{K}, u \Vdash [C]_{\Gamma}^{\top}$ . By  $\mathcal{K}, w \Vdash \Box C$  we get  $\Phi_u \vdash_{\top} C$  and then by conjunctive closure, there is some  $E \in \Phi_u \cap \Gamma$  such that  $\top \vdash E \rightarrow C$ . Then  $\top \vdash E \rightarrow C$  and  $C \Vdash_{\Gamma}^{\top} [C]_{\Gamma}^{\top}$  implies  $\top \vdash E \rightarrow [C]_{\Gamma}^{\top}$ . Since  $\top$  is sound for  $\mathcal{K}$ , compatibility condition implies  $\mathcal{K}, u \Vdash [C]_{\Gamma}^{\top}$ .

Right-to-left: Let  $u \sqsubset w$  and  $\mathcal{K}, u \Vdash [C]_{\Gamma}^{\top}$  seeking to show  $\Phi_u \vdash_{\top} C$ . From  $\mathcal{K}, u \Vdash [C]_{\Gamma}^{\top}$  and soundness of  $\top$  we get  $\mathcal{K}, u \Vdash C$ . Then  $A$ -fullness implies  $\Phi_u \vdash_{\top} C$ .  $\square$

The following theorem states that if we have some requirements, preservativity semantics coincides with usual Kripke semantics for intuitionistic modal logics.

**Theorem 4.5.** *Let  $A \in \mathcal{L}_\square$  and  $\mathcal{K} = (W, \preceq, \sqsubset, V, \varphi)$  be weakly  $A$ -full  $(\Delta, \Gamma, \top)$ -semantic and  $\mathcal{K} \Vdash \top$ . Then  $\mathcal{K}, w \Vdash B$  iff  $\tilde{\mathcal{K}}, w \Vdash B$  for every  $B \in \text{sub}(A)$  and  $w \in W$ .*

*Proof.* We use induction on  $B \in \text{sub}(A)$  and show:

$$\mathcal{K}, u \Vdash B \quad \text{iff} \quad \tilde{\mathcal{K}}, u \Vdash B.$$

Hence by induction hypothesis, for  $C \in \text{sub}(B)$  with  $C \neq B$ , and every  $v \in W$  we have

$$\mathcal{K}, v \Vdash C \quad \text{iff} \quad \tilde{\mathcal{K}}, v \Vdash C.$$

If  $B \in \text{atom}$ , obviously we are done. Also if  $B = C \circ D$  with  $\circ \in \{\vee, \wedge, \rightarrow\}$  one may use induction hypothesis to prove desired result. So it remains only  $B = \square C$ . First assume that  $\mathcal{K}, u \Vdash \square C$  and hence for every  $v \sqsubset u$  we have  $\Phi_v \vdash_\top C$ . Then by soundness of  $\top$  and compatibility condition we have  $\mathcal{K}, v \Vdash C$  and hence by induction hypothesis  $\tilde{\mathcal{K}}, v \Vdash C$  and thus  $\tilde{\mathcal{K}}, u \Vdash \square C$ . For other direction around, let  $\tilde{\mathcal{K}}, u \Vdash \square C$ . For arbitrary  $v \sqsubset u$ , we have  $\tilde{\mathcal{K}}, v \Vdash^+ C$  and then by induction hypothesis  $\mathcal{K}, v \Vdash^+ C$ . Then by weak  $A$ -fullness we have  $\Phi_v \vdash_\top C$  and thus  $\mathcal{K}, u \Vdash \square C$ .  $\square$

The forcing relationship for finite Kripke models of intuitionistic (modal) logic are obviously decidable, i.e.  $\mathcal{K}, w \Vdash A$  is decidable. The following theorem shows that such decidability holds also for some preservativity semantics.

**Theorem 4.6.** *Forcing relationship for finite  $(\Delta, \Gamma, \top)$ -semantic is decidable whenever  $(\Delta, \top)$  is recursively downward compact.*

*Proof.* Let  $\mathcal{K} = (W, \preceq, \sqsubset, V, X)$  be a  $(\Delta, \Gamma, \top)$ -semantic. We show decidability of  $\mathcal{K}, w \Vdash A$  by double induction first on  $w$  ordered by  $\sqsubset$  and second on  $A$ . So as first induction hypothesis assume that for every  $u \sqsubset w$  and every  $B \in \mathcal{L}_\square$  we have decidability of  $\mathcal{K}, u \Vdash B$ . Also as second induction hypothesis assume that for every  $u \succ w$  we have decidability of  $\mathcal{K}, u \Vdash B$  for every  $B$  which is strict subformula of  $A$ . In the following cases for  $A$ , we show decidability of  $\mathcal{K}, w \Vdash A$ :

- $A$  is atomic. Obvious.
- $A$  is a conjunction, disjunction or implication. Use the second induction hypothesis.
- $A = \square B$ . It is enough to decide  $\Delta_u \vdash_\top \varphi_u \rightarrow B$  for every  $u \sqsubset w$ . Since  $(\Delta, \top)$  is recursively downward compact, one may effectively compute  $[\varphi_u \rightarrow B]_\Delta^\top$ . By definition of  $[\cdot]_\Gamma^\top$  it is enough to decide  $\Delta_u \vdash_\top [\varphi_u \rightarrow B]_\Delta^\top$  which is equivalent to  $\mathcal{K}, u \Vdash [\varphi_u \rightarrow B]_\Delta^\top$ . Now the first induction hypothesis implies decidability of  $\mathcal{K}, u \Vdash [\varphi_u \rightarrow B]_\Delta^\top$ .  $\square$

As one expect, preservativity relation  $\Vdash_\Gamma^\top$  must be sound for preservativity semantics:

**Lemma 4.7.**  *$\Vdash_\Gamma^\top$  is sound for  $(\Delta, \Gamma, \top)$ -semantics, i.e. given such preservativity semantics  $\mathcal{K}$ , we have  $\mathcal{K} \Vdash A \triangleright B$  whenever  $A \Vdash_\Gamma^\top B$ .*

*Proof.* Let  $A \Vdash_\Gamma^\top B$  and  $\mathcal{K} = (W, \preceq, \sqsubset, V, \varphi)$  be a  $(\Delta, \Gamma, \top)$ -semantics and  $w \sqsubset u \in W$  and  $E \in \Delta$  such that  $\Phi_u, E \vdash_\top A$ . Hence there is a finite set  $\Phi'_u \subseteq \Phi_u \cap \Delta$  such that  $\Phi'_u, E, \varphi_u \vdash A$ . By conjunctive closure condition, we have  $\bigwedge \Phi_u \wedge E \wedge \varphi_u \in \Gamma$  and thus by  $A \Vdash_\Gamma^\top B$  we get  $\Phi'_u, E, \varphi_u \vdash_\top B$ . Hence we have  $\Phi_u, E \vdash_\top B$ .  $\square$



## 4.1 General soundness and completeness

In the current subsection we prove general soundness and completeness theorems of  $\text{iGLH}(\Gamma, \mathbb{T})$  for preservativity semantics.

**Theorem 4.8 (Soundness).**  *$\text{iGLH}(\Gamma, \mathbb{T})$  is sound for  $(\Delta, \Gamma, \mathbb{T})$ -semantics whenever  $\text{IPC} \subseteq \mathbb{T}$  and  $\text{SN}(\Box) \subseteq \Delta$ .*

*Proof.* Let  $\mathcal{K} = (W, \preceq, \sqsubset, V, \varphi)$  is a  $(\Delta, \Gamma, \mathbb{T})$ -semantics. We use induction on  $w \in W$  ordered by  $\sqsubset$  and show  $\mathcal{K}, w \Vdash \text{iGLH}(\Gamma, \mathbb{T})$ . We also use another induction on the proof  $\text{iGLH}(\Gamma, \mathbb{T}) \vdash A$  and show  $\mathcal{K}, w \Vdash A$ . Remember that (section 2.4)  $\text{iGLH}(\Gamma, \mathbb{T})$ 's only inference rule is modus ponens and instead of necessitation we added  $\Box A$  for every axiom instance of  $\text{iGLH}(\Gamma, \mathbb{T})$ .

- $A = (\Box(B \rightarrow C) \wedge \Box B) \rightarrow \Box C$ . Let  $v \succ w$  such that  $\mathcal{K}, v \Vdash \Box(B \rightarrow C)$  and  $\mathcal{K}, v \Vdash \Box B$ . Given  $u \sqsubset v$ , we must show  $\Phi_u \vdash_\mathbb{T} C$ . By  $\mathcal{K}, v \Vdash \Box(B \rightarrow C)$  and  $\mathcal{K}, v \Vdash \Box B$  we have  $\Phi_u \vdash_\mathbb{T} B \rightarrow C$  and  $\Phi_u \vdash_\mathbb{T} B$ . Thus  $\Phi_u \vdash_\mathbb{T} C$ .
- $A = \Box B \rightarrow \Box \Box B$ . Let  $v \succ w$  such that  $\mathcal{K}, v \Vdash \Box B$  and assume that  $u \sqsubset v$ . It is enough to show  $\Phi_u \vdash_\mathbb{T} \Box B$ . For every  $u' \sqsubset u$  by transitivity we have  $u' \sqsubset v$  and hence by  $\mathcal{K}, v \Vdash \Box B$  we get  $\Phi_{u'} \vdash_\mathbb{T} B$ . Thus  $\mathcal{K}, u \Vdash \Box B$  and since  $\Box B \in \text{SN}(\Box) \subseteq \Delta$  we have  $\Box B \in \Phi_u$ .
- $A = p \rightarrow \Box p$  for  $p \in \text{par}$ . Let  $v \succ w$  such that  $\mathcal{K}, v \Vdash p$  and  $u \sqsubset v$  seeking to show  $\Phi_u \vdash_\mathbb{T} p$ . Since  $\mathcal{K}, v \Vdash p$  we get  $\mathcal{K}, u \Vdash p$  and since  $p \in \text{SN}(\Box) \subseteq \Delta$  we have  $p \in \Phi_u$ .
- $A = \Box(\Box B \rightarrow B) \rightarrow \Box B$ . Let  $v \succ w$  such that  $\mathcal{K}, v \Vdash \Box(\Box B \rightarrow B)$  and assume that  $u \sqsubset v$  seeking to show  $\Phi_u \vdash_\mathbb{T} B$ . Since  $u \sqsubset v$ , transitivity implies that also we have  $\mathcal{K}, u \Vdash \Box(\Box B \rightarrow B)$ . First induction hypothesis implies  $\mathcal{K}, u \Vdash \Box(\Box B \rightarrow B) \rightarrow \Box B$ . Hence  $\mathcal{K}, u \Vdash \Box B$  and since  $\Box B \in \text{SN}(\Box) \subseteq \Delta$  we get  $\Box B \in \Phi_u$ . Since  $\mathcal{K}, v \Vdash \Box(\Box B \rightarrow B)$  we have  $\Phi_u \vdash_\mathbb{T} \Box B \rightarrow B$  and thus  $\Phi_u \vdash_\mathbb{T} B$ .
- $A = \Box B \rightarrow \Box C$  and  $B \overset{\mathbb{T}}{\approx} C$ . Lemma 4.7.
- $A = \Box B$  and  $B$  is an instance of above axioms. Let  $u \sqsubset w$  seeking to show  $\Phi_u \vdash_\mathbb{T} B$ . By the first induction hypothesis we have  $\mathcal{K}, u \Vdash \Box B$  and since  $\Box B \in \text{SN}(\Box) \subseteq \Delta$  we have  $\Box B \in \Phi_u$  and hence  $\Phi_u \vdash_\mathbb{T} B$ .
- $A$  is an IPC-valid theorem. Remark 4.2.
- $A = \Box B$  and  $B$  is an IPC-valid theorem. Let  $u \sqsubset w$ . Then since  $\mathbb{T} \supseteq \text{IPC}$  we have  $\mathbb{T} \vdash B$  and hence  $\Phi_u \vdash_\mathbb{T} B$ .
- $A$  is derived by modus ponens from  $B$  and  $B \rightarrow A$ . Use the second induction hypothesis.  $\square$

**Theorem 4.9 (Completeness).**  *$\text{iGLH}(\Gamma, \mathbb{T})$  is complete for good  $(\Gamma, \mathbb{T})$ -semantics, if  $(\Gamma, \mathbb{T})$  is strong downward compact and  $\Gamma \supseteq \text{SN}(\Box)$  is closed under conjunction and  $\mathbb{T} \supseteq \text{IPC}$ .*

*Proof.* Let  $\text{iGLH}(\Gamma, \mathbb{T}) \not\vdash A$ . Then by lemma 2.18 we also have  $\text{iGLH}(\Gamma, \mathbb{T}) \not\vdash \llbracket A \rrbracket_\Gamma^\mathbb{T}$  and hence  $\text{iGL} \not\vdash \llbracket A \rrbracket_\Gamma^\mathbb{T}$ . Theorem 2.2 implies that there is some good Kripke model  $\tilde{\mathcal{K}} := (W, \preceq, \sqsubset, V)$  such that  $\tilde{\mathcal{K}}, w_0 \not\vdash \llbracket A \rrbracket_\Gamma^\mathbb{T}$ . Define  $\varphi(w) := \mathbb{T}$  for the preservativity semantic  $\mathcal{K} := (W, \preceq, \sqsubset, V, \varphi)$ . The definition of  $X$  is such that evidently  $\mathcal{K}$  is compatible. Theorem 4.5 implies  $\mathcal{K}, w_0 \not\vdash \llbracket A \rrbracket_\Gamma^\mathbb{T}$  and theorem 4.8 implies  $\mathcal{K} \Vdash \text{iGLH}(\Gamma, \mathbb{T})$ . By lemma 2.18 we have  $\text{iGLH}(\Gamma, \mathbb{T}) \vdash A \leftrightarrow \llbracket A \rrbracket_\Gamma^\mathbb{T}$  and thus  $\mathcal{K}, w_0 \not\vdash A$ .  $\square$

**Corollary 4.10.**  *$\text{iGLH}_\sigma$  ( $\text{iGLC}_3\text{H}_\sigma$ ) is sound and complete for  $(\text{C}_a)$  good  $(\text{SN}, \text{iGLC}_a)$ -semantics.*

*Proof.* Theorems 3.12, 4.8 and 4.9  $\square$

**Corollary 4.11.**  $\text{iGLH}^\square$  is sound and complete for good  $(\text{SN}(\square), \text{iGL})$ -semantics.

*Proof.* Theorems 3.56, 4.8 and 4.9 □

Although it is pleasing to have such a good general soundness and completeness for several logics, there are limitations: we are not able to apply this and have completeness result for  $\text{iGLH} := \text{iGLH}(\text{C}\downarrow\text{SN}(\square), \text{iGL})$ . The major obstacle for doing so is that  $\text{C}\downarrow\text{SN}(\square)$  is not closed under conjunction. However we have the soundness result for  $\text{iGLH}$ :

**Corollary 4.12.**  $\text{iGLH}$  is sound for  $(\text{SN}(\square), \text{C}\downarrow\text{SN}(\square), \text{iGL})$ -semantics.

*Proof.* Theorem 4.8. □

## 4.2 Preservativity semantics for $\text{iGLH}$

In this section, we prove completeness of  $\text{iGLH}$  for good  $(\text{SN}(\square), \text{C}\downarrow\text{SN}(\square), \text{iGL})$ -semantics. This completeness result will be helpful in proving “ $\text{iGLH} \not\vdash A$  implies  $\text{iGLC}_a\text{H}_\sigma \not\vdash \gamma(A)$ ” (theorem 5.1) which itself implies the arithmetical completeness of  $\text{iGLH}$ . We also will use the completeness of  $\text{iGLH}$  for preservativity semantics in section 4.3 to provide decidability of  $\text{iGLH}$ .

We say that  $Y \subseteq \mathcal{L}_\square$  is  $(\Gamma, \top)$ -adequate if

- $\perp \in Y$  and  $Y$  is closed under subformulas.
- If  $A, B \in Y$  then  $\lfloor A \wedge B \rfloor_\top^\Gamma \in Y$ . More precisely, it means that for every  $A, B \in Y$  there is some  $C \in Y \cap \Gamma$  such that  $\top \vdash C \rightarrow (A \wedge B)$  and for every  $E \in \Gamma$  with  $\top \vdash E \rightarrow (A \wedge B)$  we have  $\top \vdash E \rightarrow C$ .

Also a set  $\Delta$  is called  $Y$ -saturated w.r.t.  $\top$  iff

- $\Delta \subseteq Y$ ,
- if  $\Delta \vdash_\top B$  and  $B \in Y$  then  $B \in \Delta$ ,
- $\Delta \not\vdash_\top \perp$ ,
- $B \vee C \in \Delta$  implies either  $B \in \Delta$  or  $C \in \Delta$ .

**Theorem 4.13 (Completeness).**  $\text{iGLH}$  is complete for good  $(\text{SN}(\square), \text{C}\downarrow\text{SN}(\square), \text{iGL})$ -semantics.

*Proof.* Let  $\text{iGLH} \not\vdash A$ . We construct a Kripke model  $\tilde{\mathcal{K}} := (W, \preceq, \sqsubset, V)$  as follows. By lemma 4.17 there is some finite set  $Y \ni A$  which is  $(\text{C}\downarrow\text{SN}(\square)^\vee, \text{iGL})$ -adequate. Let  $Y' := Y \cup \{\square B : B \in Y\}$  and define:

- $W :=$  the set of all  $Y'$ -saturated sets w.r.t.  $\text{iGLH}$ .
- $\preceq := \subseteq$ .
- $w \sqsubset u$  iff
  - $B, \square B \in u$ ,
  - there is some  $\square E \in u \setminus w$ .
- $w V a$  iff  $a \in w$  for atomic  $a$  and  $w \in W$ .

Before we continue with definition of  $\mathcal{K}$ , observe that  $\tilde{\mathcal{K}}$  is a Kripke model for intuitionistic modal logic and is finite irreflexive transitive  $\mathbf{C}_p$  model. Next we show that  $\tilde{\mathcal{K}} \not\models A$ . Since  $\mathbf{iGLH} \not\models A$ , lemma 4.16 implies that there is some set  $u$  which is  $Y'$ -saturated w.r.t.  $\mathbf{iGLH}$  and  $A \notin u$ . Thus by lemma 4.15 we have  $\tilde{\mathcal{K}}, u \not\models A$ . Finally we define  $\mathcal{K} := (W, \preceq, \sqsubset, V, \varphi)$  in which  $\varphi_w$  is defined for every  $w \in \sqsubset W$  as follows.

$$\varphi'_w := \bigwedge (\mathbf{C}\downarrow\mathbf{SN}(\square)^\vee \cap \bar{Y}_w) \quad \text{in which} \quad \bar{Y}_w := \{B \in Y : \mathcal{K}, w \Vdash^+ B\}$$

We will see later in this proof that  $\varphi'_w$  as defined above belongs to  $\mathbf{C}\downarrow\mathbf{SN}(\square)^\vee$ . Hence (modulo  $\mathbf{iGL}$ -provable equivalence relation)  $\varphi'_w = \bigvee \Gamma_w$  for some finite set  $\Gamma_w \subseteq \mathbf{C}\downarrow\mathbf{SN}(\square)$ . Define  $\varphi_w \in \Gamma_w$  such that  $\mathcal{K}, w \Vdash \varphi_w$ .

We will see in the rest of this proof that  $\mathcal{K}$  as defined above is a weakly  $Y$ -full  $(\mathbf{SN}(\square), \mathbf{C}\downarrow\mathbf{SN}(\square), \mathbf{iGL})$ -semantic and hence by theorem 4.5 we get  $\mathcal{K} \not\models A$ . Then lemma 4.14 implies desired completeness. So it remains only to prove by induction on  $W$  that: (1)  $\varphi'_w \in \mathbf{C}\downarrow\mathbf{SN}(\square)^\vee$  (which implies that  $\mathcal{K}$  as defined above is an  $(\mathbf{SN}(\square), \mathbf{C}\downarrow\mathbf{SN}(\square), \mathbf{iGL})$ -semantic indeed), (2)  $\Phi_w \vdash_{\mathbf{iGL}} \bar{Y}_w$ . The induction will be done on the  $\sqsubset$ -height of  $w \in W$ . The height  $h(w)$  of a node  $w \in W$  is defined as the maximum  $n$  such that the following sequence exists:

$$w_0 \sqsubset w_1 \sqsubset w_2 \sqsubset \dots \sqsubset w_n.$$

So as induction hypothesis assume that for every  $v \in W$  with  $h(v) > h(w)$  we have (1)  $\varphi'_v \in \mathbf{C}\downarrow\mathbf{SN}(\square)^\vee$ , (2)  $\Phi_v \vdash_{\mathbf{iGL}} \bar{Y}_v$ . Consider the Kripke model  $\mathcal{K}_v$  as restriction of  $\mathcal{K}$  to the nodes which are accessible (via transitive closure of  $\preceq \cup \sqsubset$ ) from  $v$ . Obviously  $\mathcal{K}, u \Vdash B$  iff  $\mathcal{K}_v, u \Vdash B$  for every accessible node  $u$  from  $v$  and every  $B$ . It is not difficult to observe that for every  $v \in W$  with  $h(v) \geq h(w)$  induction hypothesis implies  $\mathcal{K}_v$  is weakly  $Y$ -full  $(\mathbf{SN}(\square), \mathbf{C}\downarrow\mathbf{SN}(\square), \mathbf{iGL})$ -semantic. Hence by theorem 4.5 for every  $B \in Y$  and  $u$  in  $\mathcal{K}_v$  we have  $\tilde{\mathcal{K}}, u \Vdash B$  iff  $\mathcal{K}, u \Vdash B$ . Thus

(\*) for every  $B \in Y$  and every  $v$  with  $h(v) \geq h(w)$  we have  $\tilde{\mathcal{K}}, v \Vdash B$  iff  $\mathcal{K}, v \Vdash B$ .

In the rest of this proof assume that  $w \in W$  is a  $\sqsubset$ -accessible node.

- $\varphi'_w \in \mathbf{C}\downarrow\mathbf{SN}(\square)^\vee$ . We first show that if  $E, F \in \bar{Y}_w \cap \mathbf{C}\downarrow\mathbf{SN}(\square)^\vee$  then there is some  $G \in \bar{Y}_w \cap \mathbf{C}\downarrow\mathbf{SN}(\square)^\vee$  such that  $\mathbf{iGL} \vdash G \rightarrow (E \wedge F)$ . Then since  $Y$  is finite, there must be a single  $G \in \bar{Y}_w \cap \mathbf{C}\downarrow\mathbf{SN}(\square)^\vee$  such that  $G \vdash_{\mathbf{iGL}} \bar{Y}_w \cap \mathbf{C}\downarrow\mathbf{SN}(\square)^\vee$  and hence  $G$  is equivalent to  $\varphi'_w$ . Thus  $\varphi'_w \in \mathbf{C}\downarrow\mathbf{SN}(\square)^\vee$ , as required. So assume that  $E, F \in \bar{Y}_w \cap \mathbf{C}\downarrow\mathbf{SN}(\square)^\vee$ . Then (\*) implies  $\tilde{\mathcal{K}}, w \Vdash^+ E, F$ . By  $(\mathbf{C}\downarrow\mathbf{SN}(\square)^\vee, \mathbf{iGL})$ -adequacy of  $Y$ , we have  $\Box[E \wedge F] \in Y$ . Let  $w' \in W$  be such that  $w' \sqsubset w$  and  $\tilde{\mathcal{K}}, w' \Vdash \Box E \wedge \Box F$ . Lemma 4.15 implies  $\Box E, \Box F \in w'$ . Since  $\mathbf{iGLH} \vdash \Box(E \wedge F) \rightarrow \Box[E \wedge F]$ , we get  $\Box[E \wedge F] \in w'$  and hence lemma 4.15 implies  $\tilde{\mathcal{K}}, w \Vdash^+ [E \wedge F]$ . Thus (\*) implies  $\mathcal{K}, w \Vdash^+ [E \wedge F]$  and hence  $[E \wedge F] \in \bar{Y}_w \cap \mathbf{C}\downarrow\mathbf{SN}(\square)^\vee$  and  $\mathbf{iGL} \vdash [E \wedge F] \rightarrow (E \wedge F)$ .
- Weakly  $Y$ -fullness. Let  $B \in Y$  such that  $\mathcal{K}, w \Vdash^+ B$  seeking to show  $\varphi_w \vdash_{\mathbf{iGL}} B$ . Since  $\vdash \varphi_w \rightarrow \varphi'_w$ , it is enough to show  $\varphi'_w \vdash_{\mathbf{iGL}} B$ . Since  $\mathcal{K}, w \Vdash^+ B$ , by the statement (\*) we have  $\tilde{\mathcal{K}}, w \Vdash^+ B$ . By definition of  $Y$  we have  $[B] \in Y$ . Let  $w' \in W$  be such that  $w' \sqsubset w$  and  $\tilde{\mathcal{K}}, w' \Vdash \Box B$ . Then lemma 4.15 implies  $\Box B \in w'$ . Since  $\mathbf{iGLH} \vdash \Box B \rightarrow \Box[B]$  we get  $\Box[B] \in w'$  and again lemma 4.15 implies  $\tilde{\mathcal{K}}, w \Vdash^+ [B]$ . Since  $[B] \in \mathbf{C}\downarrow\mathbf{SN}(\square)^\vee$  we get  $\vdash \varphi'_w \rightarrow [B]$  and hence  $\mathbf{iGL} \vdash \varphi'_w \rightarrow B$ .  $\square$

**Lemma 4.14.** *Let  $\mathcal{K} = (W, \preceq, \sqsubset, V, \varphi)$  be a finite transitive irreflexive  $(\Delta, \Gamma, \mathbf{T})$ -model. Then for every  $w_0 \in W$  there is some good Kripke model  $\mathcal{T} = (W', \preceq', \sqsubset', V', \varphi')$  with the root  $w'_0$ , which is equivalent to  $\mathcal{K}$ , i.e. there is a function  $e : W' \rightarrow W$  such that  $e(w'_0) = w_0$  and for every  $w' \in W'$  and  $A \in \mathcal{L}_{\triangleright}$  we have  $\mathcal{T}, w' \Vdash A$  iff  $\mathcal{K}, e(w') \Vdash A$ .*

*Proof.* First define  $(W', \preceq', \sqsubset', V')$  and the function  $e$  same as in the proof of lemma 2.3. Then let  $\varphi'(w') := \varphi(e(w'))$ . The rest of proof is straightforward and left to the reader.  $\square$

**Lemma 4.15 (Truth).** *Let  $\tilde{\mathcal{K}}$  and  $Y'$  as defined in the proof of theorem 4.13. Then  $\tilde{\mathcal{K}}, w \Vdash B$  iff  $B \in w$  for every  $B \in Y'$ .*

*Proof.* We use induction on  $B \in Y'$ .

- $B \in \text{atom}$ . Then by definition we have  $B \in w$  iff  $\tilde{\mathcal{K}}, w \Vdash B$ .
- $B = E \wedge F$ . Since  $Y'$  is closed under subformulas and  $w$  is closed under derivability of propositions in  $Y'$ , we have  $E \wedge F \in w$  iff  $E, F \in w$ . Thus induction hypothesis implies desired result.
- $B = E \vee F$ . First let  $B \in w$ . Since  $w \in W$ , it is  $Y'$ -saturated w.r.t. iGLH and hence either  $E \in w$  or  $F \in w$ . Thus by induction hypothesis (and closure of  $Y$  under subformulas) we get either  $\tilde{\mathcal{K}}, w \Vdash E$  or  $\tilde{\mathcal{K}}, w \Vdash F$ . In either of the cases we have  $\tilde{\mathcal{K}}, w \Vdash E \vee F$ . For the other way around, let  $\tilde{\mathcal{K}}, w \Vdash E \vee F$ . Then either  $\tilde{\mathcal{K}}, w \Vdash E$  or  $\tilde{\mathcal{K}}, w \Vdash F$ . Hence induction hypothesis implies  $E \in w$  or  $F \in w$ . Since  $\text{iGLH} \vdash E \rightarrow (E \vee F)$  and  $\text{iGLH} \vdash F \rightarrow (E \vee F)$  we have  $E \vee F \in w$ .
- $B = E \rightarrow F$ . First assume that  $E \rightarrow F \in w$  and  $u \supseteq w$  and  $\tilde{\mathcal{K}}, u \Vdash E$  seeking to show  $\tilde{\mathcal{K}}, u \Vdash F$ . Induction hypothesis implies  $E \in u$ . Then since  $u \supseteq w$  we have  $E \rightarrow F \in u$  and thus  $F \in u$ . Again by induction hypothesis  $\tilde{\mathcal{K}}, u \Vdash F$ .

For other way around, let  $E \rightarrow F \notin w$ . Thus  $w, E \not\vdash_{\text{iGLH}} F$  and by lemma 4.16 we get some  $u \supseteq w \cup \{E\}$  which is  $Y'$ -saturated w.r.t. iGLH and  $u \not\vdash_{\text{iGLH}} F$ . Thus  $u \in W$  and  $F \notin u$  and by induction hypothesis  $\tilde{\mathcal{K}}, u \not\vdash F$  and  $\tilde{\mathcal{K}}, u \Vdash E$ . Hence  $\tilde{\mathcal{K}}, w \not\vdash E \rightarrow F$ .

- $B = \Box C$ . First assume that  $\Box C \in w$  and  $u \sqsupseteq w$  seeking to show  $\tilde{\mathcal{K}}, u \Vdash C$ . Since  $\Box C \in w$ , by definition of  $w \sqsupseteq u$ , we get  $C \in u$  and thus by induction hypothesis  $\tilde{\mathcal{K}}, u \Vdash C$ .  
For other way around, let  $\Box C \notin w$  seeking some  $u \sqsupseteq w$  such that  $\tilde{\mathcal{K}}, u \not\vdash C$ . Since  $\Box C \notin w$  we have  $w \not\vdash_{\text{iGLH}} \Box C$ . Let us define  $w_{\Box} := \{E, \Box E : \Box E \in w\}$  and  $u_{\Box} := w_{\Box} \cup \{\Box C\}$ . If  $u_{\Box} \vdash_{\text{iGLH}} C$  then  $w_{\Box} \vdash_{\text{iGLH}} \Box C \rightarrow C$  and hence  $\Box w_{\Box} \vdash_{\text{iGLH}} \Box(\Box C \rightarrow C)$ . Since  $w \vdash_{\text{iGLH}} \Box w_{\Box}$  we get  $w \vdash_{\text{iGLH}} \Box(\Box C \rightarrow C)$ . Then by Löb's principle we have  $w \vdash_{\text{iGLH}} \Box C$ , a contradiction. Hence we have  $u_{\Box} \not\vdash_{\text{iGLH}} C$ , and by lemma 4.16 there is some  $u \supseteq u_{\Box}$  which is  $Y'$ -saturated w.r.t. iGLH and  $u \not\vdash_{\text{iGLH}} C$ . Hence  $C \notin u$  and  $w \sqsupseteq u$ . Thus induction hypothesis implies  $\tilde{\mathcal{K}}, u \not\vdash C$ .  $\square$

**Lemma 4.16.** *Let  $Y$  be a set of propositions closed under subformula and  $u_0 \subseteq Y$  a set such that  $u_0 \not\vdash_{\top} A$ . Then there is some  $u \supseteq u_0$  which is  $Y$ -saturated w.r.t.  $\top$  and  $u \not\vdash_{\top} A$ .*

*Proof.* Any maximal (with respect to  $\subseteq$ ) set  $u$  of propositions with the following properties is  $Y$ -saturated with respect to  $\top$ :

- $u_0 \subseteq u \subseteq Y$ ,
- $u \not\vdash_{\top} A$ .  $\square$

**Lemma 4.17.** *For every  $A \in \mathcal{L}_{\Box}$ , there is a finite (modulo iK4-provable equivalence relation) set  $Y \ni A$  which is  $(\text{C}\downarrow\text{SN}(\Box)^{\vee}, \text{iGL})$ -adequate. Moreover one may effectively compute such finite set.*

*Proof.* We only show the existence of  $Y$  and leave its computability to the reader. First we define  $Y_0$  and  $Y_1$  and then define

$$Y' := Y_0 \cup Y_1 \cup \{B \wedge C : B \in Y_0, C \in Y_1\} \quad \text{and} \quad Y := \{\bigvee X : X \subseteq_{\text{fin}} Y'\}.$$

$Y_0$ : Let  $X := \text{sub}^{\text{atomb}}(A)$  and define

$$Y_0 := \{B \in \mathcal{L}_0(X) : c_{\rightarrow}(B) \leq c_{\rightarrow}^{\Box}(A)\}.$$

Remember that  $c_{\rightarrow}$  counts the number of nested implications which are outside boxes and  $c_{\rightarrow}^{\Box}(A)$  is defined as maximum of  $|\text{sub}^{\text{atomb}}(A)|$  and  $\max\{c_{\rightarrow}(B) : \Box B \in \text{sub}(A)\}$ . Lemma 2.1 implies that  $Y_0$  is finite up to IPC-provable equivalence relation.

$Y_1$ : Let  $Z := \text{sub}^{\text{parb}}(A) \cup \{\Box B : B \in Y_0\}$ . Then by lemma 2.8 the set  $Y'_1 := \text{NNIL}(Z)$  is finite up to iK4-provable equivalence relationship. Then define  $Y_1 := \{B^\Box : B \in Y'_1\}$ .

We show that  $Y$  is  $(\text{C}\downarrow\text{SN}(\Box)^\vee, \text{iGL})$ -adequate. Since  $Y_0$  and  $Y_1$  are closed under subformulas,  $Y$  is also closed under subformulas. Given  $B, C \in Y'$  we show that a glb for  $B \wedge C$  w.r.t.  $(\text{C}\downarrow\text{SN}(\Box)^\vee, \Gamma)$  belongs to  $Y$ . Assume that  $B = E \wedge E'$  and  $C = F \wedge F'$  with  $E, F \in Y_0$  and  $E', F' \in Y_1$ . Since  $E', F' \in \text{SN}(\Box)$ , we have  $\lfloor B \wedge C \rfloor = \lfloor E \wedge F \rfloor \wedge (E' \wedge F')$ . It is enough to show  $\lfloor E \wedge F \rfloor \in Y$ . Since  $E \wedge F \in Y_0$ , lemma 4.18 implies desired result.  $\square$

**Lemma 4.18.**  $B \in Y_0$  implies  $\lfloor B \rfloor_{\text{C}\downarrow\text{SN}(\Box)^\vee}^{\text{iGL}} \in Y$ .

*Proof.* Let  $\Pi_B^1$  as defined in definition 3.21. Lemmas 3.23 and 3.24 imply  $\Pi_B^1 \subseteq Y_0$ . On the other hand theorem 3.25 implies that  $\bigvee \Pi_B^1$  is a glb for  $B$  w.r.t.  $(\downarrow\text{N}(\Box)^\vee, \text{iGL})$ . Lemma 3.45 implies that  $\bigvee \hat{\Pi}_B^1$  is a glb for  $B$  w.r.t.  $(\text{C}\downarrow\text{SN}(\Box)^\vee, \text{iGL})$ , i.e.  $\lfloor B \rfloor_{\text{C}\downarrow\text{SN}(\Box)^\vee}^{\text{iGL}} = \bigvee \hat{\Pi}_B^1$ . As defined in the statement of lemma 3.45 we have  $\hat{\Pi}_B^1 := \{D \wedge \Box D \wedge (D^\dagger)^\Box : D \in \Pi_B^1\}$  and thus  $\lfloor B \rfloor_{\text{C}\downarrow\text{SN}(\Box)^\vee}^{\text{iGL}} \in Y$ .  $\square$

### 4.3 Decidability of iGLH

In this section we show that iGLH is decidable. The idea behind the proof of decidability is to use finite model property that we have from theorem 4.13.

So let us start with the gist: given  $A \in \mathcal{L}_\Box$  we must decide  $\text{iGLH} \vdash A$ . First compute  $n_A$  and  $\Gamma_A$ , as provided by lemma 4.20. Then check the validity of  $A$  in (root of) all good  $(\text{SN}(\Box), \text{C}\downarrow\text{SN}(\Box), \text{iGL})$ -semantics  $\mathcal{K} = (W, \preceq, \sqsubset, V, \varphi)$  with less than  $n_A$  nodes and  $\varphi(w) \in \Gamma_A$  for every  $\Box$ -accessible  $w \in W$ . Theorems 3.56 and 4.6 together with decidability of iGL imply that the validity in  $\mathcal{K}$  is decidable. If all such models satisfy  $A$  return *yes*, otherwise *no*. Thus we have following decidability result:

**Theorem 4.19.** *iGLH is decidable.*

**Lemma 4.20.** *Given  $A \in \mathcal{L}_\Box$  with  $\text{iGLH} \not\vdash A$ , there is some  $n_A \in \mathbb{N}$  and a finite set  $\Gamma_A \subseteq \text{C}\downarrow\text{SN}(\Box)$  such that:*

- $n_A$  and  $\Gamma_A$  are effectively computable,
- there is some  $(\text{SN}(\Box), \text{C}\downarrow\text{SN}(\Box), \text{iGL})$ -semantic  $\mathcal{K} = (W, \preceq, \sqsubset, V, \varphi)$  with  $\mathcal{K} \not\vdash A$  and  $|W| \leq n_A$  and  $\varphi(w) \in \Gamma_A$  for every  $\Box$ -accessible  $w \in W$ .

*Proof.* Let  $\Gamma_A := Y_A \cap \text{C}\downarrow\text{SN}(\Box)$ , in which  $Y_A$  is as provided by lemma 4.17. The number  $n_A$  and  $\mathcal{K}$  can be easily inferred from the proof of theorem 4.13.  $\square$

## 5 Provability logic of HA: arithmetical completeness

In this section we prove that iGLH (see section 4.2) is the provability logic of HA. We already have the soundness by lemma 3.51 and theorem 2.20. So it remains to prove arithmetical completeness, which is proved by propositional reduction to the completeness of  $\text{iGLC}_a\text{H}_\sigma$  for  $\Sigma_1$ -substitutions.

Historically, the Provability Logic of Peano Arithmetic, PA is discovered [Solovay, 1976] before the  $\Sigma_1$ -Provability Logic of PA [Visser, 1981]. The method used in [Visser, 1981] essentially uses Solovay's technique in [Visser, 1981]. Then [Ardeshir and Mojtahedi, 2015] shows that in a sense, the  $\Sigma_1$ -provability logic of PA is harder than the standard provability logic of PA. Later [Mojtahedi, 2021] studies *reductions* between provability logics and characterizes several other provability logics. Most notably, it is shown in [Mojtahedi, 2021] that the  $\Sigma_1$ -provability logic of HA relative in the standard model, is the hardest known provability logic. Here in this section (see theorem 5.1) we show that the  $\Sigma_1$ -provability logic of HA [Ardeshir and Mojtahedi, 2018; Visser and Zoethout, 2019] is harder than the standard provability logic of HA, or in other words, we reduce ‘‘completeness of iGLH for arithmetical interpretations in HA’’ to the ‘‘completeness if  $\text{iGLC}_a\text{H}_\sigma$  for arithmetical  $\Sigma_1$  interpretations in HA’’.

**Theorem 5.1 (Reduction).**  $\text{iGLH} \not\vdash A$  implies  $\text{iGLC}_a\text{H}_\sigma \not\vdash \theta(A)$  for some substitution  $\theta$ .

*Proof.* Let  $\text{iGLH} \not\vdash A$ . Theorem 5.4 gives us some substitution  $\gamma$  such that  $\text{iGLH}^\square \not\vdash \gamma(A)$  and hence by theorem 5.18 we have  $\text{iGLC}_a\text{H}_\sigma \not\vdash \beta(\gamma(A))$  for some substitution  $\beta$ . Then if we let  $\theta := \beta \circ \gamma$  we are done.  $\square$

**Corollary 5.2 (Arithmetical Completeness).**  $\text{iGLH}$  is complete for arithmetical interpretations in  $\text{HA}$ , i.e. if for every arithmetical substitution  $\alpha$  we have  $\text{HA} \vdash \alpha_{\text{HA}}(A)$  then  $\text{iGLH} \vdash A$ .

*Proof.* We reason contrapositively. So assume that  $\text{iGLH} \not\vdash A$  seeking some arithmetical substitution  $\alpha$  such that  $\text{HA} \not\vdash \alpha_{\text{HA}}(A)$ . By  $\text{iGLH} \not\vdash A$  and theorem 5.1 we get some propositional substitution  $\theta$  such that  $\text{iGLC}_a\text{H}_\sigma \not\vdash \theta(A)$ . Then by theorem 2.22 there is some arithmetical substitution  $\sigma$  such that  $\text{HA} \not\vdash \sigma_{\text{HA}}(\theta(A))$ . Thus  $\alpha := \sigma \circ \theta$  finishes the proof.  $\square$

Recall definitions of  $\text{iPH}$  and  $\text{iPH}^+$  from section 2.12 and define  $\text{iPH}_\square$  ( $\text{iPH}_\square^+$ ) as the  $\mathcal{L}_\square$ -fragment of  $\text{iPH}$  ( $\text{iPH}^+$ ). As a corollary to arithmetical completeness, we have:

**Corollary 5.3.**  $\text{iGLH} = \text{iPH}_\square = \text{iPH}_\square^+$ .

*Proof.* We show  $\text{iGLH} \subseteq \text{iPH}_\square \subseteq \text{iPH}_\square^+ \subseteq \text{iGLH}$ .

- $\text{iGLH} \subseteq \text{iPH}_\square$ : Lemma 3.51.
- $\text{iPH}_\square \subseteq \text{iPH}_\square^+$ : Obvious.
- $\text{iPH}_\square^+ \subseteq \text{iGLH}$ : We reason contrapositively. Assume that  $\text{iGLH} \not\vdash A$ . Then corollary 5.2 implies  $\text{HA} \not\vdash \alpha_{\text{HA}}(A)$ . Thus theorem 2.20 implies  $\text{iPH}^+ \not\vdash A$ .  $\square$

The proof of above corollary uses arithmetical soundness and completeness theorems. However the statement of  $\text{iGLH} = \text{iPH}_\square = \text{iPH}_\square^+$  tempts us for finding a propositional proof of this fact without using arithmetical interpretations. So we pose the following question:

**Question 3.** Is there a translation  $(\cdot)^t : \mathcal{L}_\triangleright \rightarrow \mathcal{L}_\square$  with the following properties:

- $\text{iPH}^+ \vdash A$  implies  $\text{iGLH} \vdash A^t$  for every  $A \in \mathcal{L}_\triangleright$ .
- $\text{iGLH} \vdash A \leftrightarrow A^t$  for every  $A \in \mathcal{L}_\square$ .

Note that if such translation exists, one may easily prove corollary 5.3 without using the arithmetical soundness-completeness theorems:

*Proof of corollary 5.3.* The proof of  $\text{iGLH} \subseteq \text{iPH}_\square \subseteq \text{iPH}_\square^+$  goes as before. For  $\text{iPH}_\square^+ \subseteq \text{iGLH}$  we have following argument. Assume that  $\text{iPH}^+ \vdash A$  for  $A \in \mathcal{L}_\square$ . Then  $\text{iGLH} \vdash A^t$  and since  $\text{iGLH} \vdash A \leftrightarrow A^t$  we get  $\text{iGLH} \vdash A$ .  $\square$

## 5.1 First step reduction: $\text{iGLH} \not\vdash A$ implies $\text{iGLH}^\square \not\vdash \gamma(A)$

In this subsection we prove theorem 5.4:  $\text{iGLH} \not\vdash A$  implies  $\text{iGLH}^\square \not\vdash \gamma(A)$ . All consequent technical lemmas are used only for the proof of theorem 5.4. For the sake of simplicity of notations in the rest of this section we use  $\vdash$  for derivability in  $\text{iGL}$ , unless we say otherwise.

For this reduction, we transfer  $(\text{SN}(\square), \text{C}\downarrow\text{SN}(\square), \text{iGL})$ -semantics to  $(\text{SN}(\square), \text{SN}(\square), \text{iGL})$ -semantics. Before we dive into long details of this transformation, let us explain some ideas behind it. Let  $\mathcal{K} \not\vdash A$  for some  $(\text{SN}(\square), \text{C}\downarrow\text{SN}(\square), \text{iGL})$ -semantics  $\mathcal{K} = (W, \preceq, \sqsubset, V, \varphi)$ . Since  $\varphi_w \in \text{C}\downarrow\text{SN}(\square)$ , there is some projective substitution  $\theta_w$  such that  $\varphi_w \xrightarrow[\text{iGL}]{\theta_w} \text{SN}(\square)$ . We use these substitutions in a systematic way to produce our intended transformation. One obstacle in this way is that, by definition of projectivity, we actually are working with the outer substitution  $\hat{\theta}_w$  and not  $\theta_w$  itself. This means that  $\hat{\theta}_w$  is identity on boxed formulas. We use the simultaneous fixed-point theorem in  $\text{iGL}$  (see section 2.13)

to simulate such condition.

**Important notation:** Unless stated otherwise, in sections 5.1 and 5.2, we use  $\vdash$  for derivability in iGL.

**Theorem 5.4.** *iGLH  $\not\vdash A$  implies iGLH $^\square \not\vdash \gamma(A)$  for some substitution  $\gamma$ .*

*Proof.* Let us first define some notions. Given a preservativity semantic  $\mathcal{K} = (W, \preceq, \sqsubset, V, \varphi)$ , we say that a node  $w \in W$  is H-node, if (1)  $w$  is  $\sqsubset$ -accessible and (2)  $\varphi_u \neq \top$  for some  $u \sqsubseteq w$ . Also define a complexity measure  $\mathfrak{d}(A)$  whenever iGLH  $\not\vdash A$ , as follows.  $\mathfrak{d}(A)$  is the minimum number  $d$  such that there is some good (SN( $\square$ ), C $\downarrow$ SN( $\square$ ), iGL)-semantic  $\mathcal{K} = (W, \preceq, \sqsubset, V, \varphi)$  with  $d$  as the number of H-nodes. Note that by theorem 4.13 there is some such preservativity semantic indeed. Then we use induction on  $\mathfrak{d}(A)$  and prove the statement of this lemma.

If  $\mathfrak{d}(A) = 0$ , then  $\mathcal{K}$  is also an (SN( $\square$ ), iGL)-semantic and hence corollary 4.11 implies iGLH $^\square \not\vdash A$ . Thus in this case we are done if we let  $\gamma :=$ identity substitution. As induction hypothesis assume that for every  $A'$  with  $\mathfrak{d}(A') < \mathfrak{d}(A)$  we have iGLH $^\square \not\vdash \gamma(A')$  for some substitution  $\gamma$ . Then by induction hypothesis, it is enough to find some substitution  $\gamma$  such that  $\mathfrak{d}(\gamma(A)) < \mathfrak{d}(A)$ . Let  $\mathcal{K}^0 = (W, \preceq, \sqsubset, V^0, \varphi^0)$  be a good (SN( $\square$ ), C $\downarrow$ SN( $\square$ ), iGL)-semantic with the root  $w_0$  such that  $\mathcal{K}^0, w_0 \not\vdash A$  and the number of H-nodes in  $\mathcal{K}^0$  is  $\mathfrak{d}(A)$ . Moreover we assume that whenever  $w \sqsubset u$ , we have  $\vdash \varphi_u^0 \rightarrow \varphi_w^0$ . The reason for such assumption follows. By compatibility condition, we have  $\mathcal{K}, w \Vdash \varphi_w^0$ . Since  $\varphi_w^0 \in \text{C}\downarrow\text{SN}(\square)$  we have  $\mathcal{K}, w \Vdash \square \varphi_w^0$  and thus  $\text{SN}(\square)_u, \varphi_u^0 \vdash \varphi_w^0$ . Hence there is some  $E \in \text{SN}(\square)_u$  such that  $\varphi_u^0, E \vdash \varphi_w^0$ . Note that  $\varphi_u^0 \wedge E \in \text{C}\downarrow\text{SN}(\square)_u$ . Thus one may change the definition of  $\varphi^0$  accordingly to satisfy mentioned extra condition.

In the rest of this subsection we fix some H-node  $w_1 \in W$  which is  $\sqsubset$ -minimal, i.e. for every  $w' \sqsubset w_1$  is not H-node. Note that whenever  $w$  is not H-node, we have  $\varphi_w^0 = \top$ . Let  $W_0 := \{w_0\}$  and  $W_{i+1} := \{w \in W : w \text{ is an immediate } \sqsubset\text{-successor of some } w' \in W_i\}$ . By an immediate  $\sqsubset$ -successor of  $w \in W$  we mean some  $u \in W$  such that  $w \sqsubset u$  and there is no  $v \in W$  with  $w \sqsubset v \sqsubset u$ . Since  $\mathcal{K}$  is conversely well-founded, there is some maximum number  $n \in \mathbb{N}$  such that  $W_n$  is nonempty. We define  $W' := \bigcup_{i=0}^n W_i$  and  $W'' := \bigcup_{i=1}^n W_i$ . In the rest of this subsection, let  $\chi := \varphi_{w_1}^0$ . Since  $\chi \in \text{C}\downarrow\text{SN}(\square)$ , there is some  $\chi$ -projective (in iGL) substitution  $\theta$  such that  $\vdash \hat{\theta}(\chi) \leftrightarrow \chi^\dagger \in \text{SN}(\square)$  and  $\vdash \chi \rightarrow (x \leftrightarrow \theta(x))$  for every variable  $x$ , and moreover  $\theta(x) = x$  for every  $x \notin \text{sub}(A)$  (by remark 3.9). Since for every  $u \sqsupset w$  we have iGL  $\vdash \varphi_u^0 \rightarrow \varphi_w^0$ , iGL  $\vdash \hat{\theta}(\chi) \leftrightarrow \chi^\dagger$  and iGL  $\vdash \chi \rightarrow (x \leftrightarrow \theta(x))$ , there is some finite set  $Y \subseteq \mathcal{L}_0(\text{parb})$  of iGL-axioms such that  $Y \vdash_{\text{IPC}} \varphi_u^0 \rightarrow \varphi_w^0$ ,  $Y \vdash_{\text{IPC}} \hat{\theta}(\chi) \leftrightarrow \chi^\dagger$  and  $Y \vdash_{\text{IPC}} \chi \rightarrow (a \leftrightarrow \theta(a))$  for every  $a \in \text{atom}$  and  $w \sqsubset u$ . Let

$$\overrightarrow{\square B} := \square B_1, \dots, \square B_m,$$

includes all boxed subformulas of elements in  $Y \cup \{\chi\}$ . Also let

$$\overrightarrow{p} := p_1, \dots, p_m$$

and  $q$  are fresh atomic parameters, i.e. they are pairwise distinct and not appeared in  $A$  or  $\theta(x)$  or  $\{\varphi_w^0, \varphi_w^\dagger\}_{w \in W''}$  for every  $x \in \text{var}$ . Also let  $\eta$  be a substitution such that  $\eta(p_i) := \square B_i$  for  $1 \leq i \leq m$  and  $\eta$  is identity elsewhere. Let  $\alpha$  be the parametric dual of  $\theta$  in the language  $\mathcal{L}_0(\text{var}, \overrightarrow{p})$ , i.e.  $\alpha$  is a substitution such that  $\eta(\alpha(B)) = \hat{\theta}(\eta(B))$  for every  $B \in \mathcal{L}_0(\text{var}, \overrightarrow{p})$ . Moreover define  $Y' \cup \{\chi'\} \subseteq \mathcal{L}_0(\text{var}, \overrightarrow{p})$  and  $\chi^\ddagger \in \mathcal{L}_0(\overrightarrow{p})$  such that  $\eta(Y') = Y$  and  $\eta(\chi^\ddagger) = \chi^\dagger$  and  $\eta(\chi') = \chi$ . Then obviously we have  $Y' \vdash_{\text{IPC}} \alpha(\chi') \leftrightarrow \chi^\ddagger$  and  $Y' \vdash_{\text{IPC}} \chi' \rightarrow (x \leftrightarrow \alpha(x))$  for every  $x \in \text{var}$ . Define the following substitution:

$$\beta(a) := \begin{cases} (q \rightarrow \alpha(a)) \wedge (\neg q \rightarrow a) & : a \in \text{var} \\ a & : a \in \text{par} \cup \{\perp\} \end{cases}$$

Let  $\tau$  be the simultaneous fixed point of  $\beta(\square B_1), \dots, \beta(\square B_m)$  with respect to  $\overrightarrow{p}$ , as provided by theorem 2.21. This means that iGL  $\vdash \tau(p_i) \leftrightarrow \tau(\beta(\square B_i))$ . Finally define  $\gamma := \tau \circ \beta$ .

Then define  $\mathcal{K}^1 := (W, \preceq, \sqsubset, V^1, \varphi^0)$  with  $W, \preceq$  and  $\sqsubset$  and  $\varphi^0$  as we had in  $\mathcal{K}^0$ , and  $V^1$  as follows:

$$w V^1 p_i \text{ iff } \mathcal{K}^1, w \Vdash \Box B_i, \quad w V^1 q \text{ iff } w_1 \sqsubseteq w \text{ iff } \exists v w_1 \sqsubseteq v \preceq w,$$

and also define  $w V^1 a$  iff  $w V^0 a$ , for every other atomic  $a$ . Note that in above definition, there is a recursion when we are defining  $V^1$  (and later when we are defining  $V^2$ ) and it is a valid definition because  $(W, \sqsubset)$  is conversely well-founded. One may easily observe by induction on  $w$  (ordered by  $\sqsubset$ ) that  $\mathcal{K}^1, w \Vdash B$  iff  $\mathcal{K}^0, w \Vdash B$  for every  $B$  that  $p_i$ 's and  $q$  not appeared. Hence  $\mathcal{K}^1$  satisfies compatibility condition and  $\mathcal{K}^1, w_0 \not\Vdash A$ . Moreover  $\mathcal{K}^1$  is also a good  $(\text{SN}(\Box), \text{C}\downarrow\text{SN}(\Box), \text{iGL})$ -semantic and  $\mathcal{K}^1$  and  $\mathcal{K}^0$  share the same set of H-nodes.

Finally define  $\mathcal{K}^2 := (W, \preceq, \sqsubset, V^2, \varphi^2)$  with  $(W, \preceq, \sqsubset)$  as we had in  $\mathcal{K}^0$  and  $(V^2, \varphi^2)$  as follows:

$$\varphi_w^2 := \begin{cases} \varphi_w^0 & : w \neq w_1 \\ \top & : w = w_1 \end{cases} \quad \text{and} \quad \begin{cases} w V^2 p_i \text{ iff } \mathcal{K}^2, w \Vdash \Box \gamma(B_i) \\ w V^2 a \text{ iff } w V^1 a \text{ for every other atomic } a \end{cases}$$

Also define  $\text{SN}(\Box)_w^i := \{E \in \text{SN}(\Box) : \mathcal{K}^i, w \Vdash E\}$  for  $i \in \{0, 1, 2\}$ . For our later applications, let us also define:

$$\psi := \bigwedge \{ \Box (p_i \leftrightarrow \Box B_i) : 1 \leq i \leq m \} \quad \text{and} \quad \psi' := \bigwedge \{ \Box (p_i \leftrightarrow \Box \gamma(B_i)) : 1 \leq i \leq m \}.$$

Note that the valuations in  $\mathcal{K}^1$  and  $\mathcal{K}^2$  are defined such that  $\mathcal{K}^1 \Vdash \psi$  and  $\mathcal{K}^2 \Vdash \psi'$ . Lemma 5.6 implies that  $\mathcal{K}^2$  is also a good  $(\text{SN}(\Box), \text{C}\downarrow\text{SN}(\Box), \text{iGL})$ -semantic and corollary 5.17 implies  $\mathcal{K}^2, w_0 \not\Vdash \gamma(A)$ . Thus  $\mathfrak{d}(\gamma(A)) < \mathfrak{d}(A)$ .  $\square$

The rest of this subsection is devoted to prove some technical lemmas required in the proof of theorem 5.4.

Let  $\mathcal{L}'_{\Box}$  indicates the set of all propositions  $B \in \mathcal{L}_{\Box}$  such that  $p_i \notin \text{sub}(B)$  for every  $p_i \in \vec{p}$ . Also let  $\Delta_1 \equiv_{\Gamma} \Delta_2$  defined as follows:

$$\forall E \in \Gamma (\Delta_1 \vdash E \Leftrightarrow \Delta_2 \vdash E).$$

Remember from theorem 4.6 that  $\Phi_w^i := \{\varphi_w^i\} \cup \text{SN}(\Box)_w^i$  for  $i \in \{0, 1, 2\}$ .

**Lemma 5.5.** *For every  $w \not\sqsubseteq w_1$  and  $E \in \mathcal{L}'_{\Box}$  we have  $\mathcal{K}^2, w \Vdash E$  iff  $\mathcal{K}^1, w \Vdash E$ . Moreover for  $\sqsubset$ -accessible  $w$  with  $w \not\sqsubseteq w_1$  we have  $\Phi_w^2 \equiv_{\mathcal{L}'_{\Box}} \Phi_w^1$  and  $\Phi_{w_1}^2 \cup \{\chi\} \equiv_{\mathcal{L}'_{\Box}} \Phi_{w_1}^1$ .*

*Proof.* The second statement is a direct consequence of first one. So we prove by a double induction, first on  $W$  ordered by  $\sqsubset$  and second on the complexity of  $E \in \mathcal{L}'_{\Box}$  and show that  $\mathcal{K}^2, w \Vdash E$  iff  $\mathcal{K}^1, w \Vdash E$ . So as first induction hypothesis for every  $u \sqsubset w$  and every  $E \in \mathcal{L}'_{\Box}$  we have  $\mathcal{K}^1, u \Vdash E$  iff  $\mathcal{K}^2, u \Vdash E$  (and hence  $\Phi_u^2 \equiv_{\mathcal{L}'_{\Box}} \Phi_u^1$ ). Also as second induction hypothesis assume that for every strict subformula  $F$  of  $E$  we have  $\mathcal{K}^1, u \Vdash F$  iff  $\mathcal{K}^2, u \Vdash F$  for every  $u \not\sqsubseteq w_1$ . We have following cases for  $E$ :

- $E \in \text{atom} \setminus \vec{p}$  or  $E = \perp$ . Obvious.
- $E$  is conjunction, disjunction or implication. All of these cases are easy consequences of second induction hypothesis.
- $E = \Box F$ . Note that by the first induction hypothesis and definitions of  $\Phi_u^1$  and  $\Phi_u^2$  we have  $\Phi_u^1 \equiv_{\mathcal{L}'_{\Box}} \Phi_u^2$  for every  $u \sqsubset w$ . Then by definition of validity for  $\Box F$  in preservativity semantics we have desired result.  $\square$

**Lemma 5.6.**  *$\mathcal{K}^2$  is a good  $(\text{SN}(\Box), \text{C}\downarrow\text{SN}(\Box), \text{iGL})$ -semantic.*

*Proof.* Lemma 5.5 implies that  $\mathcal{K}^2$  satisfies compatibility condition. Other required properties inherited from  $\mathcal{K}^1$ .  $\square$



**Lemma 5.7.**  $\psi' \vdash B \leftrightarrow \tau(B)$  for every  $B \in \mathcal{L}_\square$ .

*Proof.* Easy induction on the complexity of  $B$  and left to reader.  $\square$

**Lemma 5.8.**  $\psi, \psi', \chi, q \vdash \gamma(B) \leftrightarrow B$  and  $\psi', \neg q \vdash \gamma(B) \leftrightarrow B$ .

*Proof.* We only treat first statement for the case  $B = a \in \text{atom}$  and leave the rest to reader. By definition of  $\beta$  we have  $q \vdash_{\text{IPC}} \beta(a) \leftrightarrow \alpha(a)$ . On the other hand,  $Y', \chi' \vdash_{\text{IPC}} \alpha(a) \leftrightarrow a$  and hence  $Y', q, \chi' \vdash_{\text{IPC}} \beta(a) \leftrightarrow a$ . Then lemma 5.7 implies  $\psi', Y', q, \chi' \vdash \tau\beta(a) \leftrightarrow a$ . Since  $\psi$  implies that  $\eta$  is identity, we get  $\psi, \psi', Y, q, \chi \vdash \gamma(a) \leftrightarrow a$ . Then since  $\vdash Y$  we have desired result.  $\square$

**Lemma 5.9.**  $\Phi_w^2 \vdash \psi \wedge \psi'$  for every  $\square$ -accessible  $w \not\sqsubseteq w_1$ . Moreover  $\Phi_w^2 \vdash \gamma(B) \leftrightarrow B$  for every  $B \in \mathcal{L}_\square$  and  $\square$ -accessible  $w \not\sqsubseteq w_1$ .

*Proof.* First note that lemma 5.8 and  $\Phi_w^2 \vdash \psi \wedge \psi'$  imply  $\Phi_w^2 \vdash \gamma(B) \leftrightarrow B$  whenever  $w \not\sqsubseteq w_1$ . So we only prove the first statement by induction on  $w$  ordered by  $\sqsupseteq$ . As induction hypothesis assume that for every  $u \sqsupseteq w$  we have  $\Phi_u^2 \vdash \psi \wedge \psi'$  and hence  $\Phi_u^2 \vdash \gamma(B) \leftrightarrow B$  for every  $B \in \mathcal{L}_\square$ , seeking to show  $\Phi_w^2 \vdash \psi \wedge \psi'$ . Since  $\psi \wedge \psi' \in \text{SN}(\square)$ , it is enough to show  $\mathcal{K}^2, w \Vdash \psi \wedge \psi'$ . So it is enough to show that  $\mathcal{K}^2, w \Vdash \square(p_i \leftrightarrow \square B_i) \wedge \square(p_i \leftrightarrow \square \gamma(B_i))$ . By induction hypothesis we have  $\mathcal{K}^2, w \Vdash \square(p_i \leftrightarrow \square B_i) \wedge \square(p_i \leftrightarrow \square \gamma(B_i))$ . Also by definition of  $V'$  we have  $\mathcal{K}^2, w \Vdash p_i \leftrightarrow \square \gamma(B_i)$ . Also by induction hypothesis we have  $\mathcal{K}^2, w \Vdash \square(B_i \leftrightarrow \gamma(B_i))$  and hence  $\mathcal{K}^2, w \Vdash \square B_i \leftrightarrow \square \gamma(B_i)$ . Thus  $\mathcal{K}^2, w \Vdash p_i \leftrightarrow \square B_i$ .  $\square$

**Lemma 5.10.**  $\Phi_w^1 \vdash \psi \wedge \psi'$  for every  $\square$ -accessible  $w \not\sqsubseteq w_1$ . Moreover  $\Phi_w^1 \vdash \gamma(B) \leftrightarrow B$  for every  $B \in \mathcal{L}_\square$  and  $\square$ -accessible  $w \not\sqsubseteq w_1$ .

*Proof.* Similar to the proof of lemma 5.9 and left to reader.  $\square$

**Lemma 5.11.**  $\vdash \gamma(B) \leftrightarrow \gamma(\eta(B))$  for every  $B \in \mathcal{L}_\square$ .

*Proof.* Use induction on the complexity of  $B$ . We only treat the case  $B = p_i$  here and leave other cases to reader. Note that  $\gamma := \tau \circ \beta$  and since  $\beta$  is identity on  $p_i$  we have  $\gamma(p_i) = \tau(p_i)$ . On the other hand, since  $\tau$  is simultaneous fixed-point of  $\vec{p}$  we have  $\vdash \tau(p_i) \leftrightarrow \tau\beta(\square B_i)$ , which implies  $\vdash \gamma(p_i) \leftrightarrow \gamma(\eta(p_i))$ , as desired.  $\square$

**Lemma 5.12.** For every  $\square$ -accessible  $w \in W$  we have  $\Phi_w^2 \vdash \gamma(\varphi_w^0)$ .

*Proof.* We have following cases:

- $w \sqsupseteq w_1$ : Since  $w_1$  is  $\square$ -minimal among the H-nodes, in this case we have  $\varphi_w^0 = \top$  and hence we are trivially done.
- $w_1 \sqsupseteq w$ : In this case we have  $\Phi_w^2 \vdash \varphi_w^0$  and hence lemma 5.9 implies  $\Phi_w^2 \vdash \gamma(\varphi_w^0)$ .
- $w = w_1$ : By definition of  $\beta$  we have  $q \vdash_{\text{IPC}} \beta(x) \leftrightarrow \alpha(x)$ . On the other hand,  $Y', \chi^\dagger \vdash_{\text{IPC}} \alpha(\chi')$  and hence  $Y', q, \chi^\dagger \vdash_{\text{IPC}} \beta(\chi')$ . Then lemma 5.7 implies  $\psi', Y', q, \chi^\dagger \vdash \tau\beta(\chi')$ . Since  $\psi$  implies that  $\eta$  is identity, we get  $\psi, \psi', Y, q, \chi^\dagger \vdash \gamma(\chi')$ . Then lemma 5.11 implies  $\psi, \psi', Y, q, \chi^\dagger \vdash \gamma(\chi)$ . Also since  $\vdash Y$  and  $\Phi_{w_1}^2 \vdash q$ , we get  $\psi, \psi', \chi^\dagger \vdash \gamma(\chi)$ . On the other hand lemma 5.9 implies  $\Phi_{w_1}^2 \vdash \psi, \psi'$ . Moreover we have  $\mathcal{K}^1, w_1 \Vdash \chi^\dagger$  (because  $\mathcal{K}^1, w_1 \Vdash \chi$  and  $\text{iGL} \vdash \chi \rightarrow \chi^\dagger$ ) and hence by lemma 5.5 we have  $\mathcal{K}^2, w_1 \Vdash \chi^\dagger$ . Since  $\chi^\dagger \in \text{SN}(\square)$  by definition of  $\Phi_w^2$  we get  $\chi^\dagger \in \Phi_{w_1}^2$ . Hence we may conclude  $\Phi_{w_1}^2 \vdash \gamma(\chi)$ .
- Otherwise: By lemma 5.9 we have  $\Phi_w^2 \vdash \psi'$ . Then since  $\square \neg q \in \Phi_w^2$  and  $\Phi_w^2 \vdash \varphi_w^0$ , lemma 5.8 implies desired result.  $\square$

Let us define  $\mathcal{L}_\square''$  as the set of all propositions  $A \in \mathcal{L}_\square$  such that for every  $\square B \in \text{sub}(A)$  we have either  $B \in \mathcal{L}_0(\text{parb})$  or  $q \notin \text{sub}_o(B)$ . In other words,  $\mathcal{L}_\square''$  includes all propositions in which there is no occurrence of variables besides  $q$  other than those which occur outside of boxes.

**Lemma 5.13.** *Let  $Z \subseteq \text{SN}(\square)$  satisfies “ $Z \vdash E$  and  $E \in \text{SN}(\square)$  implies  $E \in Z$ ”. Then  $C \in \mathcal{L}''_{\square}$  and  $Z \vdash C$  implies  $\vdash E \rightarrow C$  for some  $E \in Z \cap \mathcal{L}''_{\square}$ . Also  $Z \vdash \gamma(C)$  implies  $\vdash \gamma(E) \rightarrow \gamma(C)$  for some  $E \in \text{SN}(\square) \cap \mathcal{L}''_{\square}$  with  $\gamma(E) \in Z$ .*

*Proof.* First assume that  $Z \vdash C$  seeking some  $E \in Z \cap \mathcal{L}''_{\square}$  such that  $\vdash E \rightarrow C$ . From  $Z \vdash C$  we get some  $F \in Z$  such that  $\vdash F \rightarrow C$ . Since  $F \in \text{SN}(\square)$ , by definition of  $\lfloor C \rfloor$  (we use  $\lfloor \cdot \rfloor$  for  $\lfloor \cdot \rfloor_{\text{JGL}}^{\text{SN}(\square)}$  in the rest of this proof) we get  $\lfloor C \rfloor \in Z$ . Theorem 3.55.3 implies that for every  $\square D \in \text{sub}(\lfloor C \rfloor)$  either  $\square D \in \text{sub}(C)$  or  $\square D \in \text{SN}(\square)$ . Hence by definition of  $\mathcal{L}''_{\square}$ , from  $C \in \mathcal{L}''_{\square}$  we get  $\lfloor C \rfloor \in \mathcal{L}''_{\square}$ . Thus  $E := \lfloor C \rfloor$  satisfies all required conditions.

Next assume that  $Z \vdash \gamma(C)$  seeking some  $E \in \text{SN}(\square) \cap \mathcal{L}''_{\square}$  with  $\gamma(E) \in Z$  and  $\vdash \gamma(E \rightarrow C)$ . From  $Z \vdash \gamma(C)$  we get some  $F \in Z$  such that  $\vdash F \rightarrow \gamma(C)$ . Since  $F \in \text{SN}(\square)$ , lemma 3.54 and theorem 3.57 imply  $\vdash F \rightarrow \gamma(C)^*$ ,  $\vdash \gamma(C)^* \rightarrow \gamma(C)$  and  $\text{sub}^{\text{B}}_{\circ}(\gamma(C)^*) \subseteq \text{sub}^{\text{B}}_{\circ}(\gamma(C))$ . Consider some  $\square D_0 \in \text{sub}^{\text{B}}_{\circ}(\gamma(C)^*)$ . Either we have  $\square D_0 = \gamma(\square D)$  for some  $\square D \in \text{sub}^{\text{B}}_{\circ}(C)$  or  $\square D_0 \in \text{sub}^{\text{B}}_{\circ}(\gamma(x))$  for some  $x \in \text{sub}_{\circ}(C)$ . In the latter case there exists some  $p_i \in \vec{p}$  such that  $\square D_0 = \tau(p_i)$ . Since  $\tau$  is the simultaneous fixed point, we have  $\vdash \square D_0 \leftrightarrow \gamma(\square B_i)$ . So we may conclude that there is some  $E_0 \in \text{N}(\square) \cap \mathcal{L}''_{\square}$  such that  $\vdash \gamma(C)^* \leftrightarrow \gamma(E_0)$  and by  $\vdash \gamma(C)^* \rightarrow \gamma(C)$  we have  $\vdash \gamma(E_0) \rightarrow \gamma(C)$ . On the other hand since  $\vdash F \rightarrow \gamma(E_0)$  and  $F \in \text{SN}(\square)$ , by theorem 2.5 we have  $\vdash F \rightarrow \gamma(E_0)^{\square}$ . Let  $E := E_0^{\square}$ . Then we have  $\gamma(E) \in Z$  and  $E \in \text{SN}(\square) \cap \mathcal{L}''_{\square}$ . Since  $\vdash \gamma(E_0) \rightarrow \gamma(C)$  and  $\vdash \gamma(E_0)^{\square} \rightarrow \gamma(E_0)$ , we get  $\vdash \gamma(E_0)^{\square} \rightarrow \gamma(C)$ . Since  $\gamma(E) = \gamma(E_0)^{\square}$  we have  $\vdash \gamma(E) \rightarrow \gamma(C)$ .  $\square$

**Lemma 5.14.** *Let  $B \in \mathcal{L}_0(\text{parb})$  such that for every  $\square E \in \text{sub}_{\circ}(B) \cup \{\square B_i : i \leq m\}$  and  $u \sqsupseteq w$  we have  $\Phi_u^1 \vdash E$  iff  $\Phi_u^1 \vdash \gamma(E)$ . Then  $\Phi_w^1 \vdash B \leftrightarrow \gamma(B)$ .*

*Proof.* We use induction on the complexity of  $B$ . All cases are easy except for:

- $B = p_i$  for some  $i \leq m$ . First note that by assumption of this lemma for every  $u \sqsupseteq w$  we have  $\mathcal{K}^1, u \Vdash \square(\square B_i \leftrightarrow \square \gamma(B_i))$ . In the other hand by definition of  $\mathcal{K}^1$  we have  $\mathcal{K}^1 \Vdash \square(p_i \leftrightarrow \square B_i)$ . Hence  $\mathcal{K}^1, w \Vdash \square(p_i \leftrightarrow \square \gamma(B_i))$ . Since  $\square(p_i \leftrightarrow \square \gamma(B_i)) \in \text{SN}(\square)$  we get  $\square(p_i \leftrightarrow \square \gamma(B_i)) \in \text{SN}(\square)^1_w$ . Also by definition of  $\gamma$  we have  $\gamma(p_i) = \tau(p_i)$  and then since  $\tau$  is fixed-point we have  $\vdash \tau(p_i) \leftrightarrow \square \gamma(B_i)$ . Thus  $\vdash \gamma(p_i) \leftrightarrow \square \gamma(B_i)$  which implies  $\Phi_w^1 \vdash p_i \leftrightarrow \gamma(p_i)$ .
- $B$  is a parameter other than  $p_i$ 's or  $B = \perp$ . Then  $\gamma(B) = B$  and we are trivially done.
- $B$  is a conjunction, disjunction or implication. Easy by induction hypothesis and left to the reader.
- $B = \square C$ . Then by assumption of lemma for every  $u \sqsupseteq w$  we have  $\mathcal{K}^1, u \Vdash \square(\square C \leftrightarrow \square \gamma(C))$ . Since  $\square(\square C \leftrightarrow \square \gamma(C)) \in \text{SN}(\square)$  we get  $\square(\square C \leftrightarrow \square \gamma(C)) \in \text{SN}(\square)^1_w$  and thus  $\Phi_w^1 \vdash \square C \leftrightarrow \square \gamma(C)$ .  $\square$

Lemma 5.10 states that for every  $\square$ -accessible  $w \not\sqsupseteq w_1$  and  $B \in \mathcal{L}_{\square}$  we have  $\Phi_w^1 \vdash \gamma(B) \leftrightarrow B$ . The following lemma extends a weakening of this result for  $w \sqsupseteq w_1$ .

**Lemma 5.15.** *For every  $\square$ -accessible  $w \in W$  and  $\square B \in \mathcal{L}''_{\square}$  we have  $\Phi_w^1 \vdash B$  iff  $\Phi_w^1 \vdash \gamma(B)$ . Moreover for every  $B \in \mathcal{L}''_{\square} \cap \mathcal{L}_0(\text{parb})$  we have  $\Phi_w^1 \vdash B \leftrightarrow \gamma(B)$ .*

*Proof.* The second statement is a consequence of first statement and lemma 5.14. We use induction on  $w$  ordered by  $\sqsupseteq$  and prove the first statement. So as induction hypothesis assume that for every  $\square B \in \mathcal{L}''_{\square}$  and  $u \sqsupseteq w$  we have  $\Phi_u^1 \vdash B$  iff  $\Phi_u^1 \vdash \gamma(B)$ . Observe that induction hypothesis together with lemma 5.14 imply  $\Phi_w^1 \vdash E \leftrightarrow \gamma(E)$  for very  $E \in \mathcal{L}''_{\square} \cap \mathcal{L}_0(\text{parb})$ . If  $w \not\sqsupseteq w_1$  then lemma 5.10 implies desired result. Since  $\square B \in \mathcal{L}''_{\square}$  either we have  $B \in \mathcal{L}_0(\text{parb})$  or  $q \notin \text{sub}_{\circ}(B)$ . If  $B \in \mathcal{L}_0(\text{parb})$  then by induction hypothesis we have  $\Phi_w^1 \vdash B \leftrightarrow \gamma(B)$  which implies the desired result. So we assume that  $w \sqsupseteq w_1$  and  $q \notin \text{sub}_{\circ}(B)$ .

- Assume that  $\Phi_w^1 \vdash B$  seeking to show  $\Phi_w^1 \vdash \gamma(B)$ . By  $\Phi_w^1 \vdash B$  and lemma 5.13 we get some  $E \in \Phi_w^1 \cap \mathcal{L}''_{\square}$  such that  $\vdash E \rightarrow B$ . Since  $\Phi_w^1 \subseteq \text{SN}(\square) \subseteq \mathcal{L}_0(\text{parb})$ , induction hypothesis implies  $\Phi_w^1 \vdash \gamma(E) \leftrightarrow E$ . In the other hand from  $\vdash E \rightarrow B$  we get  $\vdash \gamma(E) \rightarrow \gamma(B)$  and thus  $\vdash E \rightarrow \gamma(B)$ . Since  $E \in \Phi_w^1$  we have  $\Phi_w^1 \vdash \gamma(B)$ .

- Assume that  $\Phi_w^1 \vdash \gamma(B)$  seeking to show  $\Phi_w^1 \vdash B$ . By induction hypothesis we have  $\Phi_w^1 \vdash \gamma(B) \leftrightarrow \hat{\gamma}(B)$  and hence  $\Phi_w^1 \vdash \hat{\gamma}(B)$ . Let  $\lambda$  defined as follows:  $\lambda(q) := \perp$  and  $\lambda$  is identity elsewhere. Since iGL is closed under outer substitutions, from  $\Phi_w^1 \vdash \hat{\gamma}(B)$  we get  $\hat{\lambda}(\Phi_w^1) \vdash \hat{\lambda}(\hat{\gamma}(B))$ . Observe that by definition of  $\gamma$  we have  $\text{IPC} \vdash \hat{\lambda}(\hat{\gamma}(B)) \leftrightarrow \hat{\lambda}(B)$  and since  $q \notin \text{sub}_o(B)$  we get  $\text{IPC} \vdash \hat{\lambda}(\hat{\gamma}(B)) \leftrightarrow B$ . Hence  $\hat{\lambda}(\Phi_w^1) \vdash B$ . Obviously for every  $E \in \text{SN}(\square)$  we have  $\hat{\lambda}(E) \in \text{SN}(\square)$ . Also since  $\mathcal{K}^1, w \Vdash \neg q$  and  $\neg q \vdash E \leftrightarrow \hat{\lambda}(E)$  we get  $\Phi_w^1 \vdash \hat{\lambda}(\Phi_w^1)$  and thus  $\Phi_w^1 \vdash B$ .  $\square$

**Lemma 5.16.**  $\mathcal{K}^1, w \Vdash D$  iff  $\mathcal{K}^2, w \Vdash \gamma(D)$  for every  $D \in \mathcal{L}_0(\text{parb}) \cap \mathcal{L}''_{\square}$  and  $w \in W$ .

*Proof.* Since  $\mathcal{K}^1 \Vdash \psi$  we have  $\mathcal{K}^1 \Vdash \eta(D) \leftrightarrow D$ . Also lemma 5.11 implies  $\mathcal{K}^2 \Vdash \gamma(\eta(D)) \leftrightarrow \gamma(D)$ . Since for every  $1 \leq i \leq m$  we have  $p_i \notin \text{sub}(\eta(D))$  and  $D \in \mathcal{L}''_{\square}$  implies  $\eta(D) \in \mathcal{L}''_{\square}$ , it is enough to prove the statement of lemma with the extra assumption that  $D \in \mathcal{L}'_{\square}$ .

First by induction on  $w \in W'$ , ordered by  $\sqsupset$ , we show that for every  $D \in \mathcal{L}_0(\text{parb}) \cap \mathcal{L}'_{\square} \cap \mathcal{L}''_{\square}$ , we have  $\mathcal{K}^1, w \Vdash D$  iff  $\mathcal{K}^2, w \Vdash \gamma(D)$ . We use a second induction on the complexity of  $D$  and prove the statement of this lemma. So by first induction hypothesis we have  $\mathcal{K}^1, u \Vdash C$  iff  $\mathcal{K}^2, u \Vdash \gamma(C)$  for every  $C \in \mathcal{L}_0(\text{parb}) \cap \mathcal{L}'_{\square} \cap \mathcal{L}''_{\square}$  and  $u \sqsupset w$ . Also as second induction hypothesis we have  $\mathcal{K}^1, w' \Vdash C$  iff  $\mathcal{K}^2, w' \Vdash \gamma(C)$  for every  $w' \in W$  and  $C$  which is a strict sub-formula of  $D$ . We have following cases for  $D$ :

- $D = \perp$  or  $D$  is a parameter: Since  $D \neq p_i$  by definition of  $\gamma$  we have  $\gamma(D) = D$ . Then obviously by definitions of  $\mathcal{K}^1$  and  $\mathcal{K}^2$  we have desired result.
- $D$  is conjunction, disjunction or implication: straightforward by second induction hypothesis.
- $D = \square C$ .
  - First assume that  $\mathcal{K}^1, w \Vdash \square C$  and consider some  $u \sqsupset w$  seeking to show  $\Phi_u^2 \vdash \gamma(C)$ . By definition of  $\mathcal{K}^1, w \Vdash \square C$  we have  $\Phi_u^1 \vdash C$ . We have three cases: (1)  $u \not\sqsupseteq w_1$ . Then lemma 5.15 implies  $\Phi_u^1 \vdash \gamma(C)$  and hence lemma 5.5 implies  $\Phi_u^2 \vdash \gamma(C)$ . (2)  $u = w_1$ . Then  $\text{SN}(\square)_u \vdash \varphi_u^0 \rightarrow C$ . Moreover since  $q \notin \text{sub}(\varphi_u^0)$  we have  $\varphi_u^0 \in \mathcal{L}''_{\square}$ . Hence  $\varphi_u^0 \rightarrow C \in \mathcal{L}''_{\square}$  and lemma 5.13 implies  $\vdash E \rightarrow (\varphi_u^0 \rightarrow C)$  for some  $E \in \text{SN}(\square)_u \cap \mathcal{L}''_{\square}$ . Then  $\gamma(E), \gamma(\varphi_u^0) \vdash \gamma(C)$ . Since  $E \in \text{SN}(\square) \cap \mathcal{L}''_{\square}$ , we get  $\mathcal{K}^1, u \Vdash E$  and first induction hypothesis implies  $\mathcal{K}^2, u \Vdash \gamma(E)$  and thus  $\gamma(E) \in \Phi_u^2$ . Also lemma 5.12 implies  $\Phi_u^2 \vdash \gamma(\varphi_u^0)$  and hence  $\Phi_u^2 \vdash \gamma(C)$ . (3)  $u \sqsupset w_1$ . Then  $\Phi_u^1 \subseteq \text{SN}(\square)$  and  $\Phi_u^1 \vdash C$  together with lemma 5.13 implies  $\vdash E \rightarrow C$  for some  $E \in \Phi_u^1 \cap \mathcal{L}''_{\square}$ . Thus  $\vdash \gamma(E) \rightarrow \gamma(C)$ . Since  $E \in \Phi_u^1$  we have  $\mathcal{K}^1, u \Vdash E$  and first induction hypothesis implies  $\mathcal{K}^2, u \Vdash \gamma(E)$ . Since  $\gamma(E) \in \text{SN}(\square)$  we get  $\gamma(E) \in \Phi_u^2$  and thus  $\Phi_u^2 \vdash \gamma(C)$ .
  - For the other direction, let  $\mathcal{K}^2, w \Vdash \gamma(\square C)$  and  $u \sqsupset w$  seeking to show  $\Phi_u^1 \vdash C$ . By definition of  $\mathcal{K}^2, w \Vdash \gamma(\square C)$  we get  $\Phi_u^2 \vdash \gamma(C)$ . First assume that  $u \not\sqsupseteq w_1$ . Then  $\Phi_u^2, \varphi_u^0 \vdash \gamma(C)$  and hence lemma 5.5 implies  $\Phi_u^1 \vdash \gamma(C)$ . Then lemma 5.15 implies  $\Phi_u^1 \vdash C$ .  
Next assume that  $u \sqsupset w_1$ . From  $\Phi_u^2 \vdash \gamma(C)$  and lemma 5.13 there is some  $E \in \text{SN}(\square) \cap \mathcal{L}''_{\square}$  such that  $\gamma(E) \in \Phi_u^2$  and  $\vdash \gamma(E \rightarrow C)$ . Hence  $\mathcal{K}^2, u \Vdash \gamma(E)$  and by first induction hypothesis we get  $\mathcal{K}^1, u \Vdash E$ . Since  $E \in \text{SN}(\square)$  we have  $E \in \Phi_u^1$ . On the other hand  $\vdash \gamma(E \rightarrow C)$  implies  $\Phi_u^1 \vdash \gamma(E \rightarrow C)$  and hence  $\Phi_u^1 \vdash \gamma(E) \rightarrow \gamma(C)$ . Since  $E \in \mathcal{L}_0(\text{parb}) \cap \mathcal{L}''_{\square}$  lemma 5.15 implies  $\Phi_u^1 \vdash E \leftrightarrow \gamma(E)$  and then  $\Phi_u^1 \vdash \gamma(C)$ . Thus lemma 5.15 implies  $\Phi_u^1 \vdash C$ .  $\square$

**Corollary 5.17.**  $\mathcal{K}^1, w \Vdash D$  iff  $\mathcal{K}^2, w \Vdash \gamma(D)$  for every  $D \in \mathcal{L}''_{\square}$  and  $w \succcurlyeq w_0$ .

*Proof.* We use induction on the complexity of  $D$ :

- $D \in \text{parb}$ : Lemma 5.16.
- $D$  is atomic variable: Since  $w \succcurlyeq w_0$  we have  $w \neq w_1$  and hence  $\mathcal{K}^2, w \Vdash \neg q$  and by lemma 5.8 we have  $\mathcal{K}^2, w \Vdash \gamma(x) \leftrightarrow x$ . Then since  $\mathcal{K}^2, w \Vdash x$  iff  $\mathcal{K}^1, w \Vdash x$ , we have desired result.
- $D$  is conjunction, disjunction or implication: Easy by induction hypothesis and left to reader.  $\square$

## 5.2 Second step reduction: $\text{iGLH}^\square \not\vdash A$ implies $\text{iGLC}_a\text{H}_\sigma \not\vdash \beta(A)$

In this section we show that  $\text{iGLH}^\square \not\vdash A$  implies  $\text{iGLC}_a\text{H}_\sigma \not\vdash \beta(A)$  for some substitution  $\beta$ . The idea behind this reduction is some uniform collection of finitely many reductions of  $\text{iGL}$  to  $\text{iGLC}_a$ .

For simplicity of notations, in this subsection we use  $\lfloor \_ \rfloor$  instead of  $\lfloor \_ \rfloor_{\text{SN}(\square)}^{\text{iGL}}$ . Also we fix some  $A \in \mathcal{L}_\square$  and define:

$$\Delta := \{E \in \text{sub}(A) : E \text{ is atomic or boxed}\}, \quad \Delta' := \{E \in \text{sub}(A) : E \text{ is parameter or boxed}\}$$

$$Y := \mathcal{L}_0(\Delta), \quad Y' := \text{NNIL}(\Delta'), \quad Y'' := \{E^\square : E \in Y'\}.$$

Note that all of the above mentioned sets of propositions except for  $Y$  are finite modulo  $\text{iK4}$ -provable equivalence. Moreover lemma 3.54 and theorem 3.57 imply that for every  $E \in Y$  we have  $\lfloor E \rfloor \in Y''$ .

**Theorem 5.18.** *If  $\text{iGLH}^\square \not\vdash A$ , then there is some substitution  $\beta$  such that  $\text{iGLC}_a\text{H}_\sigma \not\vdash \beta(A)$ .*

*Proof.* Let  $\text{iGLH}^\square \not\vdash A$ . Then corollary 4.11 implies that there is some good  $(\text{SN}(\square), \text{iGL})$ -semantic  $\mathcal{K} = (W, \preceq, \sqsubset, V, \varphi)$  with the root  $w_0$  such that  $\mathcal{K}, w_0 \not\vdash A$ . Moreover we assume that for every atomic  $a \notin \text{sub}(A)$  and  $w \in W$  we have  $\mathcal{K}, w \not\vdash a$ . Let  $S := \{E : \square E \in \text{sub}(A)\}$  and  $X_w^A := \Phi_w \cap Y''$ , in which  $Y''$  is as defined just before this lemma. Note that for every  $E \in S$  we have  $\Phi_w \vdash E$  iff  $\Phi_w \vdash \lfloor E \rfloor$  iff  $\lfloor E \rfloor \in X_w^A$ . Hence  $\Phi_w \vdash E$  iff  $X_w^A \vdash E$  for  $E \in S$ . Then for every  $w \in W$  and  $E \in S$  with  $\Phi_w \not\vdash E$  let  $\mathcal{K}_E^w := (W_E^w, \preceq_E^w, \sqsubset_E^w, V_E^w)$  be some good Kripke model with the root  $w_E$  such that  $\mathcal{K}_E^w, w_E \Vdash X_w^A$  and  $\mathcal{K}_E^w, w_E \not\vdash E$ , as provided by theorem 2.2. Moreover assume that if  $(w, E) \neq (u, F)$  then  $W_E^w$  and  $W_F^u$  and  $W$  are disjoint. Also define

$$\widehat{W} := W \cup \bigcup_{E \in S, w \in W} W_E^w \quad \text{and} \quad \widehat{\sqsubset} := \sqsubset \cup \bigcup_{\Phi_w \not\vdash E} \sqsubset_E^w.$$

Also for every  $w \in \widehat{W}$  let  $\widehat{\mathcal{K}}, w \Vdash B$  indicates

- if  $w \in W$  then  $\mathcal{K}, w \Vdash B$ ,
- if  $w \in W_E^u$  then  $\mathcal{K}_E^u, w \Vdash B$ .

Let  $\{q_w : w \in \widehat{W}\}$  be a fresh set of pairwise distinct atomic parameters, i.e.  $q_u \neq q_v$  whenever  $u \neq v$  and  $q_u \notin \text{sub}(A)$  for every  $u \in \widehat{W}$ . Then define the substitution  $\beta$  as follows:

$$\beta(x) := \bigvee \{Q_w : w \in \widehat{W} \text{ and } \widehat{\mathcal{K}}, w \Vdash x\} \quad \text{and} \quad Q_w := q_w \wedge \bigwedge_{w \widehat{\sqsubset} u} \neg q_u.$$

Also define the preservativity semantic  $\bar{\mathcal{K}} := (W, \preceq, \sqsubset, \bar{V}, \bar{X})$  for  $(\text{SN}, \text{iGLC}_a)$  as follows:

- $\bar{V}$  is defined such that  $\bar{\mathcal{K}}, w \not\vdash x$  for every atomic variable  $x$  and  $\bar{\mathcal{K}}, w \Vdash q_u$  iff  $u \sqsubseteq w$  iff  $\exists v (u \sqsubseteq v \preceq w)$ . Moreover  $\bar{\mathcal{K}}, w \Vdash p$  iff  $\mathcal{K}, w \Vdash p$  for every other parameter  $p$ . Note that by this definition  $\bar{\mathcal{K}}, w \not\vdash q_u$  for every  $w \in W$  and  $u \in \widehat{W} \setminus W$ .
- $\bar{X}_w := \text{SN}(\square)'_w := \{E \in \text{SN}(\square) : \bar{\mathcal{K}}, w \Vdash E\}$ .

Similarly define  $\bar{\mathcal{K}}_E^w := (W_E^w, \preceq_E^w, \sqsubset_E^w, \bar{V}_E^w)$  in which  $\bar{V}_E^w$  is defined in this way:  $u \bar{V}_E^w a$  iff either

- $a = q_v$  such that  $u$  is accessible from  $v$  via the transitive closure of  $\preceq_E^w \cup \sqsubset_E^w$  or
- $a \in \text{sub}(A)$  is a parameter and  $u \bar{V}_E^w a$ .

Lemma 5.19 implies  $\bar{\mathcal{K}}, w \not\vdash \beta(A)$  and hence by corollary 4.10 we have  $\text{iGLC}_a\text{H}_\sigma \not\vdash \beta(A)$ .  $\square$

**Lemma 5.19.**  $\mathcal{K}, w \Vdash B$  iff  $\bar{\mathcal{K}}, w \Vdash \beta(B)$  for every  $w \in W$  and  $B \in Y$ .

*Proof.* We prove an strengthen of this lemma for arbitrary  $B \in Z := Y \cup Y''$  instead of  $B \in Y$ . We use double induction, first on  $w \in W$  ordered by  $\sqsupset$  and second on  $B \in Z$ . So as first induction hypothesis we have  $\mathcal{K}, w' \Vdash B'$  iff  $\bar{\mathcal{K}}, w' \Vdash \beta(B')$  for every  $w' \sqsupset w$  and every  $B' \in Z$ . Also as second induction hypothesis we have  $\mathcal{K}, w' \Vdash B'$  iff  $\bar{\mathcal{K}}, w' \Vdash \beta(B')$  for every  $w' \succ w$  and  $B'$  a strict subformula of  $B$ . We have the following cases for  $B \in Z$ :

- $B$  is a parameter. Since  $B \in Z$  we have  $B \neq q_w$  for every  $w \in \widehat{W}$ . Then since  $\beta(p) = p$  for every parameter  $p$ , by definition of  $\bar{V}$  in  $\bar{\mathcal{K}}$  we have desired result.
- $B = x$  is an atomic variable. Lemma 5.20.
- $B$  is conjunction, disjunction or implication: easy by first induction hypothesis and left to the reader.
- $B = \Box E$ . First assume that  $\mathcal{K}, w \Vdash \Box E$  and  $u \sqsupset w$  seeking to show  $\bar{X}_u \vdash_{\text{iGLC}_s} \beta(E)$ . From  $\mathcal{K}, w \Vdash \Box E$  we get  $\Phi_u \vdash E$  and hence  $[E] \in \Phi_u \cap Y''$ . This implies  $\mathcal{K}, u \Vdash [E]$  and first induction hypothesis implies  $\bar{\mathcal{K}}, u \Vdash \beta([E])$  and thus  $\beta([E]) \in \bar{\Phi}_u$ . Since  $\vdash [E] \rightarrow E$  we have  $\vdash \beta([E]) \rightarrow \beta(E)$  and thus  $\bar{X}_u \vdash \beta(E)$  and a fortiori  $\bar{X}_u \vdash_{\text{iGLC}_s} \beta(E)$ .

For the other direction, let  $\bar{\mathcal{K}}, w \Vdash \Box \beta(E)$  and  $u \sqsupset w$  seeking to show  $\Phi_u \vdash E$ . From  $\bar{\mathcal{K}}, w \Vdash \Box \beta(E)$  we get  $\bar{X}_u \vdash_{\text{iGLC}_s} \beta(E)$  and hence  $\bar{X}_u, \bigwedge_x \Box(x \rightarrow \Box x) \vdash \beta(E)$ . Since  $\Box(x \rightarrow \Box x) \in \bar{X}_u$ , we have  $\bar{X}_u, \bigwedge_x (x \rightarrow \Box x) \vdash \beta(E)$ . Since iGL is closed under outer substitutions, we have  $\hat{\theta}(\bar{X}_u), \hat{\theta}(\bigwedge_x (x \rightarrow \Box x)) \vdash \hat{\theta}\beta(E)$  in which  $\theta$  is the substitution which replaces  $\perp$  for every variable  $x$ . Thus  $\bar{X}_u \vdash \beta(E)$  and then there is some  $F \in \bar{X}_u$  such that  $\vdash F \rightarrow \beta(E)$ . Hence lemma 5.23 implies  $\vdash F \rightarrow \beta(G)$  and  $\vdash \beta(G) \rightarrow \beta(E)$  for some  $G \in Y''$ . Thus  $\beta(G) \in \bar{X}_u$  and then  $\bar{\mathcal{K}}, u \Vdash \beta(G)$ . Hence by second induction hypothesis we have  $\mathcal{K}, u \Vdash G$  and  $G \in X_u^A$ . Then  $\beta(X_u^A) \vdash \beta(E)$  and thus lemma 5.21 implies  $X_u^A \vdash E$  and then  $\Phi_u \vdash E$ .  $\square$

**Lemma 5.20.**  $\mathcal{K}, w \Vdash x$  iff  $\bar{\mathcal{K}}, w \Vdash \beta(x)$  for every  $w \in W$  and  $x \in \text{var}$ . Also we have  $\mathcal{K}_E^w, u \Vdash x$  iff  $\bar{\mathcal{K}}_E^w, u \Vdash \beta(x)$  for every  $u \in W_E^w$  and  $x \in \text{var}$ .

*Proof.* The proof of second statement is similar to the first one and left to reader. For the first statement we have following argument. First assume that  $\mathcal{K}, w \Vdash x$ . It is enough to show  $\bar{\mathcal{K}}, w \Vdash Q_w$ . Obviously by definition of  $\bar{\mathcal{K}}$  we have  $\bar{\mathcal{K}}, w \Vdash q_w$ . Let  $u \in \widehat{W}$  such that  $w \hat{\sqsupset} u$  seeking to show  $\bar{\mathcal{K}}, w \Vdash \neg q_u$ . Hence  $u \in W$  and  $w \sqsupset u$ . For every  $w' \succ w$  we have  $u \hat{\sqsupset} w'$ , lest  $u \sqsubseteq v \preceq w'$  and  $w \preceq w'$  implies that  $v$  and  $w$  are  $\preceq$ -comparable because  $(W, \preceq)$  is tree, and hence transcendentality implies  $v \sqsupset v$  contradicting irreflexivity. Hence  $\bar{\mathcal{K}}, w' \not\Vdash q_u$  and thus  $\bar{\mathcal{K}}, w \Vdash \neg q_u$ .

For other direction around, assume that  $\bar{\mathcal{K}}, w \Vdash \beta(x)$  seeking to show  $\mathcal{K}, w \Vdash x$ . From  $\bar{\mathcal{K}}, w \Vdash \beta(x)$  we get some  $w' \in \widehat{W}$  such that  $\bar{\mathcal{K}}, w \Vdash Q_{w'}$  and  $\hat{\mathcal{K}}, w' \Vdash x$ . From  $\bar{\mathcal{K}}, w \Vdash Q_{w'}$  we have  $\bar{\mathcal{K}}, w \Vdash q_{w'}$  and hence  $w' \sqsubseteq v \preceq w$  for some  $v \in W$  with  $\mathcal{K}, w' \Vdash x$ . If  $w' = v$  then  $w' \preceq w$  and thus  $\mathcal{K}, w \Vdash x$ . Otherwise we have  $w' \sqsupset v$  and hence  $w' \hat{\sqsupset} v$  which implies  $\bar{\mathcal{K}}, w \Vdash \neg q_v$  and thus  $\bar{\mathcal{K}}, w \not\Vdash q_v$ . On the other hand since  $v \preceq w$  we have  $v \hat{\sqsupset} w$  and hence  $\bar{\mathcal{K}}, w \Vdash q_v$ , a contradiction.  $\square$

**Lemma 5.21.**  $\beta(X_w^A) \vdash_{\text{iGLC}_s} \beta(E)$  implies  $X_w^A \vdash E$  for every  $\Box E \in \text{sub}(A)$ .

*Proof.* We reason contrapositively. Assume that  $X_w^A \not\vdash E$  seeking to show  $\beta(X_w^A) \not\vdash_{\text{iGLC}_s} \beta(E)$ . From  $X_w^A \not\vdash E$  we have  $\Phi_w \not\vdash E$  and hence  $\mathcal{K}_E^w, w_E \not\vdash X_w^A$  and  $\mathcal{K}_E^w, w_E \not\vdash E$ . Lemma 5.22 implies  $\bar{\mathcal{K}}_E^w, w_E \not\vdash \beta(X_w^A)$  and  $\bar{\mathcal{K}}_E^w, w_E \not\vdash E$  and thus  $\beta(X_w^A) \not\vdash_{\text{iGLC}_s} \beta(E)$ .  $\square$

**Lemma 5.22.** For every  $u \in W_E^w$  and  $B \in \mathcal{L}_{\Box}$  with  $\text{sub}^{\text{par}}(B) \subseteq \text{sub}^{\text{par}}(A)$  we have  $\mathcal{K}_E^w, u \Vdash B$  iff  $\bar{\mathcal{K}}_E^w, u \Vdash \beta(B)$ .

*Proof.* We prove this lemma by induction on complexity of  $B$ . All inductive steps are derived easily by induction hypothesis, except for basic steps:

- $B = x$  is an atomic variable. Lemma 5.20.

- $B = p$  is a parameter. By assumption of this lemma we must have  $p \in \text{sub}(A)$  and hence by definition of  $\bar{\mathcal{K}}_E^w$  we have  $\mathcal{K}_E^w, u \Vdash p$  iff  $\bar{\mathcal{K}}_E^w, u \Vdash p$ .  $\square$

**Lemma 5.23.** *Let  $\vdash F \rightarrow \beta(E)$  for some  $F \in \text{SN}(\square)$  and  $E \in Y \cup Y''$ . Then there is some  $G \in Y''$  such that  $\vdash F \rightarrow \beta(G)$  and  $\vdash \beta(G) \rightarrow \beta(E)$ .*

*Proof.* If  $E \in Y''$  we are trivially done by  $G := E$ . So assume that  $E \in Y$ . Assume that  $\vdash F \rightarrow \beta(E)$  seeking some  $G \in Y''$  with  $\vdash F \rightarrow \beta(G)$  and  $\vdash \beta(G) \rightarrow \beta(E)$ . Since  $F \in \text{SN}(\square)$ , lemma 3.54 and theorem 3.57 imply  $\vdash (\beta(E))^* \rightarrow \beta(E)$  and  $\vdash F \rightarrow (\beta(E))^*$  with  $(\beta(E))^* \in \text{N}(\square)$  and  $\text{sub}_o((\beta(E))^*) \subseteq \text{sub}_o(\beta(E))$ . Hence there is some  $G_0 \in \text{N}(\square)$  such that  $\beta(G_0) = (\beta(E))^*$  and  $\text{sub}_o(G_0) \subseteq \text{sub}_o(\beta(E))$  and thus  $G_0 \in Y$ . Let  $G := G_0^\square$ . Obviously we have  $G \in Y''$ . Since  $\vdash F \rightarrow \beta(G_0)$ , theorem 2.5 implies  $\vdash F \rightarrow \beta(G)$ . Moreover since  $\vdash \beta(G) \rightarrow \beta(G_0)$  and  $\vdash \beta(G_0) \rightarrow \beta(E)$  we get  $\vdash \beta(G) \rightarrow \beta(E)$ .  $\square$

## Acknowledgement

Special acknowledgement heartily granted to Amirhossein Akbar Tabatabai, Mohammad Ardeshir, Lev Beklemishev and Albert Visser for stimulating discussions or communications we had on provability logic.

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