

# Localizing Finite-Depth Kripke Models

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## Abstract

We can look at a first-order (or propositional) intuitionistic Kripke model as an ordered set of classical models. In this paper, we show that for a finite-depth Kripke model in an arbitrary first-order language or propositional language, local (classical) truth of a formula is equivalent to non-classical truth (truth in the Kripke semantics) of a Friedman's translation of that formula, i.e.  $\alpha \Vdash A^\rho \Leftrightarrow \mathfrak{M}_\alpha \models A$ . We introduce some applications of this fact. We extend the result of [AH02] and show that semi-narrow Kripke models of Heyting Arithmetic HA are locally PA.

*keywords:* Intuitionistic logic, Kripke models, local truth, finite depth.

## 1 Introduction

D. van Dalen et al. in [vDMKV86] introduced a very useful technique, called *pruning* of a Kripke model, for studying Kripke semantics of HA. Their method is a correspondence between forcing of Friedman's translation of a sentence in a Kripke model, and forcing of that sentence in a sub-model (in the sense of [Vis02]) of the same Kripke model. By this method, they proved that every finite Kripke model of HA is PA-normal, and every  $\omega$ -frame Kripke model of HA is locally PA for infinitely many nodes of the model. Then K. F. Wehmeier in [Weh96] strengthened this result to a wider class of Kripke models, e.g., finite-depth Kripke models, and some special infinite Kripke models. Ardehshir and Hesaam in [AH02] showed that every rooted narrow tree-frame Kripke model of HA is locally PA. In this paper, by iterated use of the pruning lemma introduced in [vDMKV86], we show that for any node  $\alpha$  of a finite depth Kripke model, there exists a sentence  $\rho$ , such that for all formula  $A$

$$\alpha \Vdash A^\rho \text{ if and only if } \alpha \models A,$$

where  $A^\rho$  is Friedman's translation of  $A$  by  $\rho$ .

## 2 Definitions, conventions and preliminaries

The propositional language  $\mathcal{L}_0$  contains  $\{\vee, \wedge, \rightarrow, \perp\}$  and atomic variables  $p_1, p_2, \dots$ . The language  $\mathcal{L}_1$  is the first-order language, i.e. as connectives contains  $\{\vee, \wedge, \rightarrow, \perp\}$ , and quantifiers  $\{\forall, \exists\}$ , plus some function symbols, relation symbols, a special equality symbol  $=$ , constant symbols and variables  $x, y, z, \dots$ . We use  $\neg A$  as an abbreviation for  $A \rightarrow \perp$ . The language of arithmetic  $\mathcal{L}_a$  contains  $\{+, \cdot\}$ ,  $\{=, <\}$  and  $\{0, 1\}$  as two function symbols, predicate symbols and constant symbols, respectively. For arbitrary set  $D$ , we use the notation  $\mathcal{L}(D)$  as the language  $\mathcal{L}$  augmented with the new set of constant symbols  $D$ . We use  $\vdash$  and  $\vdash^c$ , for intuitionistic and classical deductions, respectively.

A Kripke model for a language  $\mathcal{L}$  is a quadruples  $\mathcal{K} = (K, \leq, D, \Vdash)$  with the following properties:

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- $K$  is a non-empty set (of *nodes*), and  $(K, \leq)$  is a poset,
- $D$  is a function (the domain function) from  $K$  such that  $D(\alpha)$  is non-empty,
- For all  $\alpha \leq \beta \in K$ , we have  $D(\alpha) \subseteq D(\beta)$ ,
- $\Vdash$  is a binary relation with first component in  $K$  and second component in the set of atomic sentences in  $\mathcal{L}(D(\alpha))$ ,
- For all  $\alpha \leq \beta$  and atomic sentence  $A$  in  $\mathcal{L}(D(\alpha))$ , if  $\alpha \Vdash A$  then  $\beta \Vdash A$  (*monotonicity*).

We can extend  $\Vdash$  to all sentences in the language  $\mathcal{L}$  recursively, just like classical case, except for  $\rightarrow$  and  $\forall$  by the following items:

- $\alpha \Vdash A \rightarrow B$  iff for all  $\beta \geq \alpha$ , if  $\beta \Vdash A$  then  $\beta \Vdash B$ ,
- $\alpha \Vdash \forall x A$  iff for all  $\beta \geq \alpha$  and  $b \in D(\beta)$ , we have  $\beta \Vdash A[x : b]$ .

By this definition we can assign to each Kripke model  $\mathcal{K} = (K, \leq, D, \Vdash)$ , a triple  $(K, \leq, \mathcal{F})$ , where  $\mathcal{F}(\alpha) = \mathfrak{M}_\alpha$  is a classical model for the language  $\mathcal{L}(D(\alpha))$ , with  $D(\alpha)$  as its universe with the property: “for each atomic formula  $A$ ,  $\mathfrak{M}_\alpha \models A$  iff  $\alpha \Vdash A$ ”. Let  $T$  be a first-order theory in  $\mathcal{L}_a$ . A Kripke model  $\mathcal{K} = (K, \leq, D, \Vdash)$  is called *T-normal* or *locally T*, if  $\mathcal{F}(\alpha) \models T$ , for all  $\alpha \in K$ . In the rest of the paper, we use the notation  $\alpha \models A$  instead of  $\mathcal{F}(\alpha) \models A$ .

For a fixed sentence  $\rho$  and a sentence  $A$  in a language  $\mathcal{L}$ , Friedman’s translation of  $A$  by  $\rho$ ,  $A^\rho$ , is defined recursively by replacing all occurrences of atomic sub-formulas of  $A$  by their disjunction with  $\rho$ . More precisely,  $A^\rho$  is defined inductively as follows:

- $A^\rho := A \vee \rho$ , for atomic formula  $A$ ,
- $(A_1 \circ A_2)^\rho := A_1^\rho \circ A_2^\rho$  and  $\circ \in \{\vee, \wedge, \rightarrow\}$ ,
- $(\forall x A)^\rho := \forall x(A^\rho)$  and  $(\exists x A)^\rho := \exists x(A^\rho)$ .

We have the following facts about Friedman’s translation (see [vDMKV86]):

**Proposition 2.1.** •  $\rho \vdash A^\rho$ ,

- if  $\Gamma \vdash A$  then:  $\Gamma^\rho \vdash A^\rho$ ,
- $\vdash^c A^\rho \leftrightarrow (A \vee \rho)$ ,
- For any  $A \in \Sigma_1$ , we have  $\text{HA} \vdash A^\rho \leftrightarrow (A \vee \rho)$ ,
- $\text{HA} \vdash \text{HA}^\rho$  (*HA is closed under Friedman’s translation*)

**Lemma 2.2.**  $\neg\rho \vdash A \leftrightarrow A^\rho$ .

*Proof.* Use induction on the complexity of  $A$ . □

We call a node  $\alpha$  in a Kripke model  $\mathcal{K} = (K, \leq, D, \Vdash)$ , *classical*, iff  $\alpha \Vdash \forall \bar{x}(A \vee \neg A)$ , for all  $A$  in  $\mathcal{L}(D(\alpha))$ , where  $\bar{x}$  are all free variables of  $A$ . We have the following facts from [vDMKV86]:

**Proposition 2.3.** For any Kripke models  $\mathcal{K} = (K, \leq, D, \Vdash)$ ,

- the following conditions are equivalent:
  - $\alpha$  is a classical node,
  - $\alpha$  forces all  $\mathcal{L}$ -sentences  $\forall \bar{x}(A \vee \neg A)$ ,
  - For all  $A \in \mathcal{L}(D(\alpha))$ :  $\alpha \models A$  iff  $\alpha \Vdash A$ ,
- all final nodes are classical,
- if  $\alpha$  is classical, then so is  $\beta$  for all  $\beta \geq \alpha$ .

### 3 Localizing finite-depth Kripke models

Let  $\mathcal{K} = (K, \leq, D, \Vdash)$  be a Kripke model and  $\rho$  be a fixed sentence. We can define a new Kripke model, the *pruned* model with respect to  $\rho$ ,  $\mathcal{K}^\rho = (K^\rho, \leq^\rho, D^\rho, \Vdash^\rho)$ , where  $K^\rho = K \setminus \{\alpha \in K \mid \alpha \Vdash \rho\}$  and  $\leq^\rho$ ,  $D^\rho$  and  $\Vdash^\rho$  are restriction of  $\leq$ ,  $D$ , and  $\Vdash$ , respectively to the set  $K^\rho$ .

**Lemma 3.1. (Pruning Lemma)** *[vDMKV86] Let  $\rho \in \mathcal{L}$  be a sentence and  $\mathcal{K} = (K, \leq, D, \Vdash)$  be a Kripke model for the language  $\mathcal{L}$  and  $\alpha \in K$  such that  $\alpha \not\Vdash \rho$ . Then for all sentences  $A$  in the language  $\mathcal{L}(D(\alpha))$  :*

$$\alpha \Vdash^\rho A \text{ iff } \alpha \Vdash A^\rho.$$

The following lemma shows that Friedman's translations are associative:

**Lemma 3.2.** *For all sentences  $\rho_1, \rho_2$  and formula  $A$ :*

$$\vdash A^{(\rho_1^{\rho_2})} \leftrightarrow (A^{\rho_1})^{\rho_2}.$$

*Proof.* We prove this lemma by induction on the complexity of  $A$ :

- $A$  is an atomic. First note that using the first item of Proposition 2.1 implies  $\vdash \rho_2 \rightarrow \rho_1^{\rho_2}$ , and then  $\vdash (\rho_2 \vee \rho_1^{\rho_2}) \leftrightarrow \rho_1^{\rho_2}$ . Hence  $(A^{\rho_1})^{\rho_2} = A \vee \rho_2 \vee \rho_1^{\rho_2} \leftrightarrow A \vee \rho_1^{\rho_2} = A^{(\rho_1^{\rho_2})}$ .
- $A = C \circ B$  and  $\circ \in \{\vee, \wedge, \rightarrow\}$ . Then  $(C \circ B)^{(\rho_1^{\rho_2})} = C^{(\rho_1^{\rho_2})} \circ B^{(\rho_1^{\rho_2})} \leftrightarrow (C^{\rho_1})^{\rho_2} \circ (B^{\rho_1})^{\rho_2} = (C^{\rho_1} \circ B^{\rho_1})^{\rho_2} = (A^{\rho_1})^{\rho_2}$ .
- $A = QxB$  and  $Q \in \{\forall, \exists\}$ . Then

$$A^{(\rho_1^{\rho_2})} = (QxB)^{(\rho_1^{\rho_2})} = QxB^{(\rho_1^{\rho_2})} \leftrightarrow Qx(B^{\rho_1})^{\rho_2} = (QxB^{\rho_1})^{\rho_2} = (A^{\rho_1})^{\rho_2}$$

□

**Definition 3.3.** *Let  $\Gamma$  be a set of formulas. We define  $\Gamma^* = \bigcup \Gamma^n$ , in which  $\Gamma^n$  is defined inductively as follows*

- $\Gamma^0 := \{\perp\}$ ,
- $\Gamma^{n+1} := \{A^B : A \in \Gamma^n, B \in \Gamma\}$ .

*Also we define  $\text{PEM}(\mathcal{L})$  as the set of the universal closures of all instances of the principle of excluded middle  $A \vee \neg A$  in the language  $\mathcal{L}$ . Moreover,  $\text{PEM}_{\text{sen}}(\mathcal{L})$  is defined as the set of all instances of the principle of excluded middle  $A \vee \neg A$ , for sentence  $A$  in  $\mathcal{L}$ . When no confusion is likely, we might skip  $\mathcal{L}$  in the notation  $\text{PEM}(\mathcal{L})$  and other similar notations.*

Note that in the above definition  $\Gamma^1$  includes (an equivalent form of) all formulas  $A \in \Gamma$ ,  $\Gamma^2$  includes (an equivalent form of) all  $A^B$ , in which  $A, B \in \Gamma$ ,  $\Gamma^3$  includes (an equivalent form of) all  $(A^B)^C$ , in which  $A, B, C \in \Gamma$  and so on. More importantly  $\Gamma^*$  is closed under  $\Gamma$ -Friedman's translation, i.e. for all  $A \in \Gamma^*$  and  $B \in \Gamma$ , we have  $A^B \in \Gamma^*$ .

**Definition 3.4.** *1. Let  $\alpha$  be a node of a Kripke model  $\mathcal{K}$  and let  $\mathcal{K}_\alpha$  denotes the truncated of  $\mathcal{K}$  with respect to  $\alpha$ , i.e., restriction of  $\mathcal{K}$  to all nodes  $\beta \geq \alpha$ , with the same forcing relation for atomic formulas as  $\mathcal{K}$ .*

2. *We define the depth of  $\mathcal{K}$ , indicated by  $d(\mathcal{K})$ , as the maximum natural number  $n$ , such that no path in  $(K, \leq)$  is longer than  $n$ . We denote  $d_{\mathcal{K}}(\alpha) := d(\mathcal{K}_\alpha)$  or simply  $d(\alpha) := d(\mathcal{K}_\alpha)$  if no confusion is likely.*

3. We also define  $\mathcal{K}'_\alpha$  as the restriction of the nodes of  $\mathcal{K}$  to the following set:

$$\{\alpha\} \cup \{\beta : \beta > \alpha \text{ is not classical}\} \cup \{\beta : \beta > \alpha \wedge \neg\exists\gamma(\gamma > \beta)\},$$

with the same forcing relation for atomic formulas as  $\mathcal{K}$ . In other words,  $\mathcal{K}'_\alpha$  is derived from  $\mathcal{K}_\alpha$  by eliminating all classical nodes which are strictly above  $\alpha$  and are not leaves.

Now we have our main result.

**Theorem 3.5.** *Suppose  $\mathcal{K} = (K, \leq, D, \Vdash)$  is a finite-depth Kripke model for the language  $\mathcal{L}$ . Then for any  $\alpha \in K$ , there exists some  $\rho \in \text{PEM}(\mathcal{L})^*$  such that for any sentences  $A$  in  $\mathcal{L}(D(\alpha))$ ,*

$$\alpha \Vdash A^\rho \text{ iff } \alpha \models A.$$

*Proof.* We use induction on  $d(\alpha)$ .

- If  $d(\alpha) = 0$ , then  $\alpha$  is terminal node (a leaf) and hence by Proposition 2.3, it is a classical node. Then we take  $\rho := \perp \in \text{PEM}(\mathcal{L})^0$ .
- Suppose that we have the induction hypothesis for all  $\mathcal{K} = (K, \leq, D, \Vdash)$ ,  $\beta \in K$  with  $d(\beta) < n$ . Let  $\mathcal{K} = (K, \leq, D, \Vdash)$  a finite-depth Kripke model,  $\alpha \in K$  and  $d(\alpha) = n > 0$ . If  $\alpha$  is a classical node, by Proposition 2.3, we may let  $\rho := \perp$ . Otherwise, Proposition 2.3 implies  $\alpha \not\models \forall\bar{x}(A(\bar{x}) \vee \neg A(\bar{x}))$ , for some formula  $A(\bar{x}) \in \mathcal{L}$  with free variables in  $\bar{x}$ . Let  $\tau := \forall\bar{x}(A(\bar{x}) \vee \neg A(\bar{x})) \in \text{PEM}$ . Then by Pruning Lemma, for any  $B$ ,  $\alpha \Vdash^\tau B$  iff  $\alpha \Vdash B^\tau$ . By Proposition 2.3, we know that  $d(K_\alpha^\tau) < n$ , and by induction hypothesis, there exists some  $\rho' \in \text{PEM}(\mathcal{L})^*$  such that for all  $A$ , we have

$$\alpha \Vdash^\tau A^{\rho'} \text{ iff } \alpha \models A \text{ iff } \alpha \Vdash (A^{\rho'})^\tau.$$

By associativity of Friedman's translation (Lemma 3.2), we have

$$\alpha \models A \text{ iff } \alpha \Vdash A^{(\rho')^\tau}.$$

Now we define  $\rho := \rho'^\tau$ . Since  $\rho' \in \text{PEM}^* = \bigcup_{i \in \mathbb{N}} \text{PEM}^i$ , there is some  $k \in \mathbb{N}$  such that  $\rho' \in \text{PEM}^k$ . Hence by Definition 3.3  $\rho = \rho'^\tau \in \text{PEM}^{k+1} \subseteq \text{PEM}^*$ , as desired.  $\square$

The above theorem could be adopted for the propositional language as well.

**Theorem 3.6.** *Let  $\mathcal{K} = (K, \leq, \Vdash)$  be a finite-depth Kripke model for the propositional language. For any  $\alpha \in K$ , there exists some  $\rho \in \text{PEM}_{\mathcal{L}_0}^*$  such that for any proposition  $A$ ,*

$$\alpha \Vdash A^\rho \text{ iff } \alpha \models A.$$

**Remark 3.7.** A sentence  $\rho \in \mathcal{L}$  is called a *localizer* for some node  $\alpha$  of a Kripke model for the language  $\mathcal{L}$ , if for any sentence  $A \in \mathcal{L}(D(\alpha))$ ,

$$\alpha \Vdash A^\rho \iff \alpha \models A.$$

In the next proposition, we show that it is *not* possible to find some localizer  $\rho$  to be applied *uniformly* for all Kripke models and nodes with some given height. This means that  $\rho$  really depends on the Kripke model and the assigned node.

**Proposition 3.8.** *Given some number  $d \geq 1$  and a first-order language  $\mathcal{L}$ , it is not possible to find some localizer  $\rho$  for all  $\alpha$  in an arbitrary Kripke model with  $d(\alpha) = d$ .*

*Proof.* We prove by contradiction. For the sake of contradiction, assume some uniform localizer  $\rho$ , for all nodes with depth  $d$ .

*Claim:*  $\vdash^c \neg\rho$ .

Before we continue with the proof of the claim, let us see how this claim finishes the proof. From the claim one can deduce that  $\rho$  is not forced in the leaves of any Kripke model (since in leaves intuitionistic and classical truth coincide). Hence  $\neg\rho$  is forced in any node of any finite-depth Kripke model. Lemma 2.2 implies that for any node  $\alpha$  of any finite-depth Kripke model and for all sentence  $A$ , we have  $\alpha \Vdash A^\rho \leftrightarrow A$ . This implies that  $\alpha \Vdash A$  iff  $\alpha \Vdash A^\rho$ . Since for any  $\alpha$  with the depth  $d$ , we have  $\alpha \Vdash A^\rho$  iff  $\alpha \models A$ , one may deduce  $\alpha \Vdash A$  iff  $\alpha \models A$ . Then it is quite straightforward to find some Kripke model  $\mathcal{K} = (K, \leq, D, \Vdash)$ ,  $\alpha \in K$  with  $d(\alpha) = d \geq 1$  and  $A \in \mathcal{L}(D(\alpha))$ , such that it is not the case that  $\alpha \Vdash A$  iff  $\alpha \models A$ . This contradicts our previous result.

*Proof of the claim:* Assume that  $\not\vdash^c \neg\rho$ . Then there exists some classical structure  $\mathfrak{M} \models \rho$ . Define a Kripke model  $\mathcal{K}$  by adding  $d - 1$  copies of  $\mathfrak{M}$  in beneath of  $\mathfrak{M}$ . Let  $\alpha_0$  be the root of  $\mathcal{K}$ . Then clearly  $d(\alpha_0) = d$  and hence for any  $A \in \mathcal{L}(D(\alpha_0))$ , we have  $\alpha_0 \Vdash A^\rho$  iff  $\alpha_0 \models A$ . Since  $\mathfrak{M} \models \rho$ , we have  $\alpha_0 \Vdash A^\rho$  for any  $A \in \mathcal{L}(D(\alpha_0))$ . Then for all  $A \in \mathcal{L}(D(\alpha_0))$ , we have  $\mathfrak{M} \models A$ . In particular,  $\mathfrak{M} \models \perp$ , a contradiction.  $\square$

What happens for infinite-depth Kripke models? In this case, there might not exist any localizer at all. Here we will present a counter-example for the propositional language. Since the propositional language is a special case of a first-order language, this counter-example is a counter-example for the first-order language as well. Let  $\mathcal{K}$  be any Kripke model for which the propositional intuitionistic logic is complete (for example the canonical model is such a Kripke model). Add some node  $\alpha_0$  in beneath of all other nodes of  $\mathcal{K}$ . From completeness of  $\mathcal{K}$  for the intuitionistic propositional logic, we have  $\alpha_0 \Vdash A$  iff  $\vdash A$ , for any  $A$ . We will show that  $\alpha_0$  doesn't have any localizer. Suppose not, i.e.  $\rho$  is a localizer for  $\alpha_0$ . Then  $\alpha_0 \Vdash A^\rho$  iff  $\alpha_0 \models A$ , for any  $A$ . Let  $A = \perp$ . Hence  $\alpha_0 \not\vdash \rho$ . By soundness, we have  $\not\vdash \rho$ . Since for atomic  $A$ , we have  $A^\rho = A \vee \rho$ , and by disjunction property of intuitionistic (propositional) logic, and  $\not\vdash \rho$ , we may deduce  $\alpha_0 \models A$  iff  $\vdash A$  for atomic  $A$ . Hence  $\alpha_0 \models A \rightarrow \perp$ , for all atomic  $A$ . Then for all atomic  $A$ , we have  $\vdash A \rightarrow \rho$ . This implies that  $\vdash \rho$ , a contradiction.

**Remark 3.9.** *Localizers for infinite-depth nodes of Kripke models might not exist.*

Although localizers for infinite depth Kripke models may not exist, we will show that, by use of methods in [AH02], for a class of semi-narrow Kripke models (definition comes next), which includes finite-depth and also some infinite models, there exist some sort of localizers (Theorem 3.14).

**Definition 3.10.** *A Kripke model is narrow if there is no infinite set of pairwise incomparable nodes. We say that a Kripke model is semi-narrow, if for any set of pairwise incomparable nodes  $X$  there is some  $n$  such that for almost all  $u \in X$  (all but finitely many of them), we have  $d(u) \leq n$ .*

Note that all finite depth Kripke models and also all narrow Kripke models are semi-narrow, but the converse is not necessarily true. For example the comb frame is semi-narrow and it is neither narrow nor finite-depth.

In [AH02], it is shown that all rooted narrow Kripke models of HA are locally PA. Here we extend that result to the class of semi-narrow models and also show that they have some sort of localizers.

For a Kripke model  $\mathcal{K} = (K, \leq, D, \Vdash)$  and  $X \subseteq K$ , let  $r_1(\mathcal{K}, X) := \#\{\alpha \in X : d(\alpha) \text{ is infinite}\}$  (the operator  $\#$  counts the cardinality of its operand) and  $r_2(\mathcal{K}, X) := \max\{d(\alpha) + 1 : \alpha \in X \text{ and } d(\alpha) \text{ is finite}\}$  and  $r(\mathcal{K}, X) := (r_1(\mathcal{K}, X), r_2(\mathcal{K}, X))$ . Finally define

$$r(\mathcal{K}) := \max\{r(\mathcal{K}, X) : X \text{ is a set of pairwise incomparable nodes in } \mathcal{K}\}$$

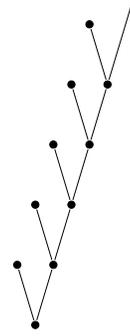
in which we use  $<$  as lexicographical order on pairs of numbers. Through these definitions, as is common, we assume that  $\max\{\} := 0$ . We say  $r(\mathcal{K})$  is finite if its both components are finite. Note that  $\mathcal{K}$  is semi-narrow iff  $r(\mathcal{K})$  is a finite number. In the above examples, the rank for “tick”, “V” and “comb” frames are  $(1, 4)$ ,  $(2, 0)$  and  $(1, 1)$ , respectively.



“tick” frame



“V” frame



“comb” frame

**Lemma 3.11.** *Let  $\mathcal{K} = (K, \leq, D, \Vdash)$  be a rooted semi-narrow Kripke model (with  $\alpha_0$  as its root) for the language  $\mathcal{L}$ . Also let  $\rho = A \vee \neg A$  be a sentence in the language  $\mathcal{L}(D(\alpha_0))$  such that  $\mathcal{K} \not\models \rho$ . Then  $r(\mathcal{K}^\rho) < r(\mathcal{K})$ .*

*Proof.* One may easily prove the lemma by observing the following facts:

1.  $X \subseteq K^\rho$  is pairwise incomparable in  $\mathcal{K}^\rho$  iff it is pairwise incomparable in  $\mathcal{K}$ ,
2.  $d_{\mathcal{K}^\rho}(\alpha) < d_{\mathcal{K}}(\alpha)$ , for any  $\alpha \in K^\rho$ ,
3. for any set  $X \subseteq K$  of pairwise incomparable nodes, we have

$$r(\mathcal{K}^\rho, X \cap K^\rho) < r(\mathcal{K}, X)$$

□

We say that  $\alpha \in K$  is *weakly classical* in  $\mathcal{K} = (K, \leq, D, \Vdash)$  if  $\alpha \Vdash \text{PEM}_{\text{sen}}(\mathcal{L}(D(\alpha)))$ .

**Lemma 3.12.** *Let  $\mathcal{K} = (K, \leq, D, \Vdash)$  be a semi-narrow Kripke model with tree frame for the language  $\mathcal{L}$ . Then for any  $\alpha \in K$ , there exists some  $\rho \in \text{PEM}_{\text{sen}}(\mathcal{L}(D(\alpha)))^*$  such that  $\alpha$  is weakly classical in  $\mathcal{K}^\rho$ .*

*Proof.* Without loss of generality, we may assume that  $\alpha$  is the root of  $\mathcal{K}$ . We use induction on  $r(\mathcal{K})$  and prove the lemma. As induction hypothesis, assume that for any Kripke model  $\mathcal{K}_1 := (K_1, \leq_1, D_1, \Vdash_1)$  with  $r(K_1) < n$ , the lemma holds and let  $\mathcal{K}$  be a rooted Kripke model with  $r(\mathcal{K}) = n$ , and  $\alpha$  as its root. If for any sentence  $A$  in the language  $\mathcal{L}(D(\alpha))$ , it holds that  $\mathcal{K}, \alpha \Vdash A \vee \neg A$ , then  $\alpha$  is weakly classical and  $\rho := \perp$  works. So assume that  $\mathcal{K}, \alpha \not\models A \vee \neg A$ , for some sentence  $A \in \mathcal{L}(D(\alpha))$ . Let  $\delta := A \vee \neg A$ . By Lemma 3.11, we have  $r(\mathcal{K}^\delta) < r(\mathcal{K})$  and induction hypothesis applies to  $\mathcal{K}^\delta$ . Hence there exists some  $\theta \in \text{PEM}_{\text{sen}}(\mathcal{L}(D(\alpha)))^*$  such that  $(\mathcal{K}^\delta)^\theta$  is weakly classical at  $\alpha$ . Since  $(\mathcal{K}^\delta)^\theta = \mathcal{K}^{\theta^\delta}$ , and  $\rho := \theta^\delta \in \text{PEM}_{\text{sen}}(\mathcal{L}(D(\alpha)))^*$ , we have the desired result. □

Let us define the translation  $(A)^\forall$  from [AH02]. For a formula  $A$  in an arbitrary language, let  $A^\forall$  be the formula obtained from  $A$  by replacing any  $\forall x B$  subformula of  $A$  by  $\forall x \neg \neg B$  (This is a variant of the Kuroda translation [TvD88, 3.3.7]).

The following lemma is from [AH02].

**Lemma 3.13.** *Let  $\mathcal{K} = (K, \leq, D, \Vdash)$  be a Kripke model and  $\alpha$  be a weakly classical node. Then for any sentence  $A$  in  $\mathcal{L}(D(\alpha))$ ,  $\alpha \Vdash A^\forall$  iff  $\alpha \models A$ .*

*Proof.* Use induction on the complexity of  $A$ . □

**Theorem 3.14.** For a semi-narrow Kripke model  $\mathcal{K} = (K, \leq, D, \Vdash)$  with tree frame for a language  $\mathcal{L}$  and any  $\alpha \in K$ , there exists some  $\rho \in \text{PEM}_{\text{sen}}(\mathcal{L}(D(\alpha)))^*$  such that for all sentences  $A \in \mathcal{L}(D(\alpha))$ ,

$$\alpha \Vdash (A^\forall)^\rho \quad \text{iff} \quad \alpha \models A$$

*Proof.* Use Lemmas 3.1, 3.12 and 3.13. □

## 4 Refinements

In this section we strengthen Theorem 3.5. We will examine the question whether is it possible to minimize the set PEM in Theorem 3.5? In Theorem 4.5, we will show that  $\text{PEM}_1$  (see Definition 4.1) is enough, however we do not know if  $\text{PEM}_1$  is the minimal set.

Hosoi in [Hos67] introduces slices  $\mathcal{S}_n$  for the intermediate logics and Ono in [Ono71] shows that there is a tight relationship between slices and depth- $n$  Kripke models in the following sense. “The logic of a Kripke frame is in the slice  $\mathcal{S}_n$  iff the height of the Kripke frame is  $n$ ”.

In this paper, we use the height (depth) of Kripke models to slice the formulas in the language.

**Definition 4.1.** Let  $\mathcal{L}$  be an arbitrary first-order language or propositional language. The Kripke-rank of a formula  $A \in \mathcal{L}$ ,  $h_{\mathcal{L}}(A)$ , is the minimum number  $n$ , such that there exists some depth- $n$  Kripke model refuting  $A$ ,  $\mathcal{K} \not\models A$ . If there is some infinite-depth Kripke model which refutes  $A$  and no finite-depth Kripke model refuting  $A$ , then we define  $h(A) := \omega$ . If there is no Kripke model  $\mathcal{K} \not\models A$ , we define  $h(A) := \infty$ . For a set of formulas  $\Gamma \subseteq \mathcal{L}$ , let  $\Gamma_n := \{A \in \Gamma : h_{\mathcal{L}}(A) = n\}$ .

It is clear that in any language ( $\sqcup$  means disjoint union)

$$\text{PEM} = \text{PEM}_\infty \sqcup \text{PEM}_\omega \sqcup \bigsqcup_{k \in \mathbb{N}} \text{PEM}_k$$

Since intuitionistic propositional logic has finite model property, there is no  $A \in \mathcal{L}_0$  with  $h(A) = \omega$ . Hence  $\text{PEM}_\omega = \emptyset$  in propositional language. Before we continue, let’s observe that  $\text{PEM}_n(\mathcal{L}_0)$  is nonempty, for any  $n \in \omega$ . Define  $A_n \in \mathcal{L}_0$  by the following clauses:

- $A_0 := \perp$ ,
- $A_{n+1} := p_n \vee (p_n \rightarrow A_n)$ .

**Proposition 4.2.** For all  $n \in \mathbb{N}$ , we have  $A_n \vee \neg A_n \in \text{PEM}_n(\mathcal{L}_0)$ .

*Proof.* First we show that  $h(A_n) = n$ , by induction on  $n$ . We note that the same Kripke model which refutes  $A_n$  also refutes  $A_n \vee \neg A_n$ . This implies the desired result. □

In the following lemma, we use the notation  $\mathcal{K}'_\alpha$  from Definition 3.4, and  $\mathcal{K}'_\alpha, \alpha \Vdash A$  means that the node  $\alpha$  in model  $\mathcal{K}'_\alpha$  forces  $A$ .

**Lemma 4.3.** Let  $\alpha$  be a node of the finite-depth Kripke model  $\mathcal{K}$ . Then for all  $A \in \mathcal{L}(D(\alpha))$ ,

$$\mathcal{K}'_\alpha, \alpha \Vdash A \quad \text{iff} \quad \mathcal{K}, \alpha \Vdash A.$$

*Proof.* Proof is by induction on  $d(\alpha)$ .

- $d(\alpha) = 0$ . In this case,  $\mathcal{K}'_\alpha = \mathcal{K}_\alpha$ .
- $d(\alpha) = n > 0$ . Note that for all non-classical  $\beta > \alpha$ , we have  $\mathcal{K}'_\beta = (\mathcal{K}'_\alpha)_\beta$ . Then for all  $A \in \mathcal{L}(D(\beta))$ ,

$$\mathcal{K}'_\alpha, \beta \Vdash A \quad \text{iff} \quad \mathcal{K}'_\beta, \beta \Vdash A.$$

This, by induction hypothesis, implies

$$\mathcal{K}'_\alpha, \beta \Vdash A \text{ iff } \mathcal{K}, \beta \Vdash A.$$

Also it's not difficult to observe that for any classical node  $\beta > \alpha$ , there exists some leaf  $\gamma \geq \beta$  (actually any leaf above  $\beta$  works) such that for all  $A \in \mathcal{L}(D(\beta))$ , we have

$$\mathcal{K}, \gamma \Vdash A \text{ iff } \mathcal{K}, \beta \Vdash A.$$

By use of the above mentioned facts, it is routine to prove the result by induction on  $A \in \mathcal{L}(D(\alpha))$ . □

**Proposition 4.4.** *For all finite-depth Kripke models  $\mathcal{K} = (K, \leq, D, \Vdash)$ ,*

$$\alpha \text{ is a classical node iff } \alpha \Vdash \text{PEM}_1.$$

*Proof.* Left to right direction is deduced by Proposition 2.3. For the other way around, we use induction on  $d(\alpha)$ .

- $d(\alpha) = 0$ . That is obvious.
- $d(\alpha) = n > 0$ . Since  $\alpha \Vdash \text{PEM}_1$ , for all  $\beta > \alpha$ , we have  $\beta \Vdash \text{PEM}_1$ , and by induction hypothesis  $\beta$  is classical node. This implies that  $d(\mathcal{K}'_\alpha) = 1$  and hence no PEM instance could be refuted in  $\mathcal{K}'_\alpha$  other than those which are in  $\text{PEM}_1$ . This implies that  $\mathcal{K}'_\alpha \Vdash \text{PEM}$ . Lemma 4.3 implies that  $\mathcal{K}, \alpha \Vdash \text{PEM}$  and then by Proposition 2.3, we can deduce that  $\alpha$  is classical. □

**Theorem 4.5.** *Suppose  $\mathcal{K} = (K, \leq, D, \Vdash)$  is a finite-depth Kripke model for the language  $\mathcal{L}$ . For any  $\alpha \in K$ , there exists some  $\rho \in \text{PEM}_1^*$  such that for any sentence  $A$  in  $\mathcal{L}(D(\alpha))$ ,*

$$\alpha \Vdash A^\rho \text{ iff } \alpha \models A.$$

*Proof.* The same proof of Theorem 3.5 works here, by using Proposition 4.4 and replacing PEM by  $\text{PEM}_1$ . □

## 5 Some applications

Now we state some applications of Theorem 4.5:

**Corollary 5.1.** *Let  $T$  be a theory which is closed under Friedman's translation  $(\cdot)^\rho$ , for any  $\rho \in \text{PEM}_1$ . Then any finite-depth Kripke model of  $T$  is locally  $T$ .*

*Proof.* First note that by iterated use of Lemma 3.2,  $T$  is closed under Friedman's translation  $(\cdot)^\rho$ , for any  $\rho \in \text{PEM}_1^*$ . Let  $\mathcal{K} = (K, \leq, D, \Vdash)$  be a finite-depth Kripke model for  $T$  and  $\alpha \in K$ . By Theorem 4.5, we can find a sentence  $\rho \in \text{PEM}_1^*$  such that for each sentence  $\varphi$  in the language  $\mathcal{L}(D(\alpha))$ , we have

$$\alpha \Vdash \varphi^\rho \iff \alpha \models \varphi$$

Since  $\alpha \Vdash T$  and  $T \vdash T^\rho$ , then  $\alpha \Vdash T^\rho$ . Hence  $\alpha \models T$ . □

Now we can deduce a result first appeared in [vDMKV86]:

**Remark 5.2.** *HA is closed under arbitrary Friedman's translation, by proposition 2.1, hence every finite-depth Kripke model of HA is locally PA.*

**Corollary 5.3.** *Let  $T$  be a theory over the language  $\mathcal{L}$  which is closed under the translation  $()^\forall$  and PEM-Friedman's translation. Then any semi-narrow Kripke model of  $T$  with tree frame is locally  $T$ .*

*Proof.* Let  $\mathcal{K}, \alpha \Vdash T$  be a semi-narrow Kripke model with tree frame. By Theorem 3.14, there is some  $\rho \in \text{PEM}(\mathcal{L}(D(\alpha)))^*$  such that for all  $A \in \mathcal{L}(D(\alpha))$ , we have  $\alpha \Vdash (A^\forall)^\rho$  iff  $\alpha \models A$ . Since  $\alpha \Vdash T$  and  $T$  is closed under PEM-Friedman's translation and  $()^\forall$  translation, we have  $\alpha \Vdash (T^\forall)^\rho$ . Hence  $\alpha \models T$ .  $\square$

**Remark 5.4.** Since HA is closed under PEM-Friedman's translation and  $()^\forall$  translation ([AH02]), we can deduce from the above Corollary that all semi-narrow Kripke models of HA are locally PA.

For any sets  $\Gamma$  and  $\Delta$  of formulas in  $\mathcal{L}$ , let  $\Gamma^\Delta := \{A^B : A \in \Gamma, B \in \Delta\}$ . Then we have:

1.  $\Gamma^{\Delta^*}$  is closed under the  $\Delta$ -Friedman's translation, i.e. for any  $A \in \Gamma^{\Delta^*}$  and  $B \in \Delta$ ,  $A^B$  is intuitionistically equivalent to some formula in  $\Gamma^{\Delta^*}$ ,
2.  $\Gamma^{\Delta^*}$  is the closure of  $\Gamma$  under the  $\Delta$ -Friedman's translation, i.e.  $\Gamma^{\Delta^*}$  is the minimum set  $X \supseteq \Gamma$  such that for all  $A \in X$  and  $B \in \Delta$  we have  $A^B \in X$ ,
3.  $i\Gamma^{\Delta^*}$  is closed under  $\Delta$ -Friedman's translation.

The first item can be deduced easily by use of Lemma 3.2, and the third item is a consequence of the first one. Second item is straightforward. We have the following facts:

$$(5.1) \quad i\Gamma \subseteq i\Gamma^{\text{PEM}_1^*} \subseteq i\Gamma^{\text{PEM}^*} \subseteq i\Gamma^{\text{Sen}^*} = i\Gamma^{\text{Sen}}$$

in which Sen is the set of all sentences in the language of arithmetic  $\mathcal{L}_a$ . Since for every set  $\Gamma$  of formulas,  $\perp \in \Gamma^*$ , if we let  $\Gamma$  as the set of all formulas in  $\mathcal{L}_a$ , all above theories are the same and equal to HA.

**Question 5.5.** *In case  $\Gamma = \Sigma_n$ ,  $\Gamma = \Pi_n$  or  $\Gamma = \Phi_n$  (definition comes next), are the inclusions of eq. (5.1) strict?*

Let us recall that  $\text{PA}^-$  indicates the set of axioms for non-negative discretely ordered ring as stated in [Kay91]. Let  $\Gamma$  be an arbitrary set of formulas. Then  $i\Gamma$  ( $i\Gamma$ ) is the  $\vdash^c$ -closure ( $\vdash$ -closure) of  $\text{PA}^-$  plus induction principle for arbitrary formulas in  $\Gamma$ . PA and HA are  $\mathcal{I}\mathcal{L}_a$  and  $i\mathcal{L}_a$ , respectively. The Burr's classes  $\Phi_n$  of formulas in  $\mathcal{L}_a$  ([Bur00]), are defined inductively by the following items:

- $\Phi_0 := \{A \in \mathcal{L}_a : A \text{ is open}\}$ ,
- $\Phi_1 := \{\exists \bar{x} A : A \in \Phi_0\}$ , ( $\bar{x}$  means a list of variables)
- $\Phi_n := \{\forall \bar{x}(B \rightarrow \exists \bar{y}C) : B \in \Phi_{n-1}, C \in \Phi_{n-2}\} \cup \Phi_{n-1}$ , for  $n \geq 2$ .

Some interesting facts about Burr's classes of formulas are

- Every formula in  $\mathcal{L}_a$  is equivalent (in  $i\Sigma_1$  and even weaker theories) to a formula in some  $\Phi_n$ ,
- For  $n \geq 2$ , every formula in  $\Phi_n$  is classically equivalent to some  $\Pi_n$  formula,
- For every  $n \geq 2$ ,  $\text{III}_n$  is  $\Pi_2$ -conservative over  $i\Phi_n$ .

These properties make the Burr's fragments  $i\Phi_n$  as *natural* fragments of HA.

**Corollary 5.6.** *Burr's hierarchies of HA,  $i\Phi_n$  are not closed under  $\text{PEM}_1$ -Friedman's translation, i.e. there exists a formula  $A$  such that  $i\Phi_n \vdash A$  but  $i\Phi_n \not\vdash A^\rho$  and  $\rho \in \text{PEM}_1$ .*

*Proof.* From [Pol06], we know that for each  $n$ , we can find a finite Kripke model for  $i\Phi_n$  such that it is not locally a model of  $i\Phi_n$ . Now by the previous corollary, we have the desired result.  $\square$

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