Hard Provability Logics

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Dedicated to Mohammad Ardeshir, for 17 years of his impact, inspirations and motivations.

Abstract

Let $\mathcal{PL}(\mathsf{T},\mathsf{T}')$ and $\mathcal{PL}_{\Sigma_1}(\mathsf{T},\mathsf{T}')$ respectively indicates the provability logic and Σ_1-provability logic of T relative in T'. In this paper we characterize the following relative provability logics: $\mathcal{PL}_{\Sigma_1}(\mathsf{HA},\mathbb{N})$, $\mathcal{PL}_{\Sigma_1}(\mathsf{HA},\mathsf{PA})$, $\mathcal{PL}_{\Sigma_1}(\mathsf{HA}^*,\mathbb{N})$, $\mathcal{PL}_{\Sigma_1}(\mathsf{HA}^*,\mathsf{PA})$, $\mathcal{PL}(\mathsf{PA},\mathsf{HA})$, $\mathcal{PL}_{\Sigma_1}(\mathsf{PA},\mathsf{HA})$, $\mathcal{PL}(\mathsf{PA}^*,\mathsf{HA})$, $\mathcal{PL}_{\Sigma_1}(\mathsf{PA}^*,\mathsf{HA})$, $\mathcal{PL}(\mathsf{PA}^*,\mathsf{PA})$, $\mathcal{PL}_{\Sigma_1}(\mathsf{PA}^*,\mathsf{PA})$, $\mathcal{PL}(\mathsf{PA}^*,\mathbb{N})$, $\mathcal{PL}_{\Sigma_1}(\mathsf{PA}^*,\mathbb{N})$ (see Table 6). It turns out that all of these provability logics are decidable.

The notion of *reduction* for provability logics, first informally considered in [AM15]. In this paper, we formalize a generalization of this notion (Definition 4.1) and provide several reductions of provability logics (See diagram 7). The interesting fact is that $\mathcal{PL}_{\Sigma_1}(\mathsf{HA},\mathbb{N})$ is the hardest provability logic: the arithmetical completenesses of all provability logics listed above, as well as well-known provability logics like $\mathcal{PL}(\mathsf{PA},\mathsf{PA})$, $\mathcal{PL}(\mathsf{PA},\mathbb{N})$, $\mathcal{PL}_{\Sigma_1}(\mathsf{PA},\mathsf{PA})$, $\mathcal{PL}_{\Sigma_1}(\mathsf{PA},\mathbb{N})$ and $\mathcal{PL}_{\Sigma_1}(\mathsf{HA},\mathsf{HA})$ are all propositionally reducible to the arithmetical completeness of $\mathcal{PL}_{\Sigma_1}(\mathsf{HA},\mathbb{N})$.

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1 Dedication

My works in general and the present paper in particular are highly inspired by Mohammad Ardeshir's outstanding contributions to mathematical logic. The ideas that I developed in this paper originate from a joint paper [AM15] which was initially motivated by him. My first recollection of Mohammad Ardeshir goes back to 2003 when, in the second semester of my undergraduate studies, I attended his course on the foundations of mathematics. I was impressed by his knowledge and by the style of his teaching which encouraged me to attend most of his other courses during my undergraduate and graduate studies. I still vividly remember how deeply I was fascinated by his graduate course on Gödel's incompleteness theorems in Fall 2006. It was this course which made me determinate to do my PhD on mathematical logic and under the supervision of Mohammad Ardeshir. His influence on me is not restricted to my academic work. He has been a source of inspiration on many aspects of my life; and that is why dedicating this paper to him is the least thing I can do to thank him.

2 Introduction

There are two excellent surveys on provability logic: [BV06, AB04]. To be self-contained, we bring some selected subjects from them here, and then review some related recent results on this subject.

The provability interpretation for the modal operator $\Box$, first considered by Kurt Gödel [Göd33], intending to provide a semantic for Heyting's formalization of the intuitionistic logic, IPC. On the other hand, and again by innovative and celebrated Gödel's incompleteness results [Göd31], for a recursively enumerable theory T and a sentence in the language of T, one may formalize "A is provable in T" via a simple (Σ_1) formula $\mathsf{Prov}_\mathsf{T}(\ulcorner A \urcorner)$ in the first-order language of arithmetic, in which $\ulcorner A \urcorner$ is the Gödel number of A. Let $\mathcal{PL}(\mathsf{T},\mathsf{T}')$ and $\mathcal{PL}_{\Sigma_1}(\mathsf{T},\mathsf{T}')$ respectively indicates the provability logic and Σ_1-provability logic of T relative in T' (Definition 3.2). Here is a list of results on provability logics with arithmetical flavour:

1. $\neg\Box\bot \notin \mathcal{PL}(\mathsf{PA},\mathsf{PA})$, [Göd31]
2. $\Box(\Box A \to A) \to \Box A \in \mathcal{PL}(\mathsf{PA},\mathsf{PA})$, [L5̈5]
3. $A \in \mathcal{PL}(\mathsf{HA},\mathsf{HA})$ for a nonmodal proposition A, iff A is valid in the intuitionistic logic IPC. [dJ70, dJVV11]
4. $\mathsf{GL} = \mathcal{PL}(\mathsf{PA},\mathsf{PA})$ and $\mathsf{GL\underline{S}} = \mathcal{PL}(\mathsf{PA},\mathbb{N}) = \mathcal{PL}(\mathsf{PA},\mathsf{ZF})$, [Sol76], in which GL is the Gödel-Löb logic, as defined in Definition 3.4.
5. $\Box(A \vee B) \to (\Box A \vee \Box B) \notin \mathcal{PL}(\mathsf{HA},\mathsf{HA})$, [Myh73, Fri75]
6. $\Box(A \vee B) \to \Box(\boxdot A \vee \boxdot B) \in \mathcal{PL}(\mathsf{HA},\mathsf{HA})$, in which $\boxdot A$ is a shorthand for $A \wedge \Box A$, [Lei75]
7. $\mathsf{iGLCT} = \mathcal{PL}(\mathsf{PA}^*,\mathsf{PA}^*)$, [Vis81, Vis82], in which iGLCT is as defined in Definition 3.4,
8. $\Box\neg\neg\Box A \to \Box\Box A \in \mathcal{PL}(\mathsf{HA},\mathsf{HA})$ and $\Box(\neg\neg\Box A \to \Box A) \to \Box(\Box A \vee \neg\Box A) \in \mathcal{PL}(\mathsf{HA},\mathsf{HA})$, [Vis81, Vis82]
9. Rosalie Iemhoff 2001 introduced a uniform axiomatization of all known axiom schemas of $\mathcal{PL}(\mathsf{HA},\mathsf{HA})$ in an extended language with a bimodal operator $\rhd$. In her Ph.D. dissertation [Iem01], Iemhoff raised a conjecture that implies directly that her axiom system, iPH, restricted to the normal modal language, is equal to $\mathcal{PL}(\mathsf{HA},\mathsf{HA})$, [Iem01]
10. $\mathcal{PL}_{\{\top,\bot\}}(\mathsf{HA},\mathsf{HA})$ is decidable. [Vis02]. In other words, he introduced a decision algorithm for $A \in \mathcal{PL}(\mathsf{HA},\mathsf{HA})$, for all A not containing any atomic variable.
11. $\mathcal{PL}_{\Sigma_1}(\mathsf{HA},\mathsf{HA}) = \mathsf{iH}_\sigma$ (Definition 3.28) is decidable, [AM18, VZ19]
12. $\mathcal{PL}_{\Sigma_1}(\mathsf{HA}^*,\mathsf{HA}^*) = \mathsf{iH}^*_\sigma$ (Definition 3.28) is decidable, [AM19]

As it is known in the literature [TvD88], the Heyting Arithmetic HA, enjoys disjunction property: if $\mathsf{HA} \vdash A \vee B$, then either $\mathsf{HA} \vdash A$ or $\mathsf{HA} \vdash B$. Regrettably, HA is not able to prove this [Fri75, Myh73]. Hence, such properties, are not reflected in the provability logic of HA, as a valid principle $\Box(A \vee B) \to (\Box A \vee \Box B)$. A natural question arises here: *is there any other valid rule?*
One way to systematically answer this question, is to characterise the truth provability logic of HA. In the case of classical arithmetic PA, Robert Solovay in his original innovative paper [Sol76], characterized the truth provability logic of PA. He showed that the only extra valid axiom is the soundness principle $\Box A \to A$, which is known to be true and unprovable in PA. In this paper we show that, in the Σ_1-provablity logic of HA, the same thing happens: The truth Σ_1-provability logic of HA, is a decidable and only has the extra axiom schema $\Box A \to A$. The disjunction property, which we mentioned before, will be deuced from Leivant's principle $\Box(A \vee B) \to \Box(\Box A \vee \Box B)$ and the soundness principle.

The author of this paper in his joint paper with Mohammad Ardeshir [AM15], showed that the arithmetical completeness of the modal logic GL, is reducible to the arithmetical completeness of $\mathsf{GL} + p \to \Box p$ for Σ_1 interpretations. The reduction involves only propositional argument. In this paper, we show that all relative provability logics, discussed in this paper, are reducible to the truth Σ_1-provability logic of HA (see Diagram 7). So, in a sense, $\mathcal{PL}_{\Sigma_1}(\mathsf{HA},\mathbb{N})$ is the hardest among them.

With the handful propositional reductions, we will characterize several relative provability logics for HA, PA, HA^* and PA^*, the self-completion of HA and PA [Vis82].

3 Definitions and Preliminaries

The propositional non-modal language $\mathcal{L}_0$ contains atomic variables, $\vee, \wedge, \to, \bot$ and the propositional modal language, $\mathcal{L}_\Box$ has an additional operator $\Box$. In this paper, the atomic propositions (in the modal or non-modal language) include atomic variables and $\bot$. For an arbitrary proposition A, $\mathsf{Sub}(A)$ is defined to be the set of all sub-formulae of A, including A itself. We take $\mathsf{Sub}(X) := \bigcup_{A\in X}\mathsf{Sub}(A)$ for a set of propositions X. We use $\boxdot A$ as a shorthand for $A \wedge \Box A$. The logic IPC is intuitionistic propositional non-modal logic over the usual propositional non-modal language. The theory $\mathsf{IPC}_\Box$ is the same theory IPC in the extended language of the propositional modal language, i.e. its language is the propositional modal language and its axioms and rules are same as IPC . Because we have no axioms for $\Box$ in $\mathsf{IPC}_\Box$, it is obvious that $\Box A$ for each A, behaves exactly like an atomic variable inside $\mathsf{IPC}_\Box$. First-order intuitionistic logic is denoted IQC and the logic CQC is its classical closure, i.e. IQC plus the principle of excluded middle. For a set of sentences and rules $\Gamma \cup \{A\}$ in the propositional non-modal, propositional modal or first-order language, $\Gamma \vdash A$ means that A is derivable from Γ in the system $\mathsf{IPC}, \mathsf{IPC}_\Box, \mathsf{IQC}$, respectively. For an arithmetical formula, $\ulcorner A \urcorner$ represents the Gödel number of A. For an arbitrary arithmetical theory T with a $\Delta_0(\exp)$-set of axioms, as far as we work in strong enough theories which is the case in this paper, we have the $\Delta_0(\exp)$-predicate $\mathsf{Proof}_\mathsf{T}(x, \ulcorner A \urcorner)$, that is a formalization of "x is the code of a proof for A in T". Note that by (inspection of the proof of) Craig's theorem, every recursively enumerable theory has a $\Delta_0(\mathsf{exp})$-axiomatization. We also have the provability predicate $\mathsf{Prov}_\mathsf{T}(\ulcorner A \urcorner) := \exists x\, \mathsf{Proof}_\mathsf{T}(x, \ulcorner A \urcorner)$. The set of natural numbers is denoted by $\omega := \{0, 1, 2, \ldots\}$.

Definition 3.1. Suppose T is a $\Delta_0(\mathsf{exp})$-axiomatized theory and σ is a substitution i.e. a function from atomic variables to arithmetical sentences. We define the interpretation σ_T which extend the substitution σ to all modal propositions A, inductively:

- $\sigma_\mathsf{T}(A) := \sigma(A)$ for atomic A,
- σ_T distributes over $\wedge, \vee, \to$,
- $\sigma_\mathsf{T}(\Box A) := \mathsf{Prov}_\mathsf{T}(\ulcorner \sigma_\mathsf{T}(A) \urcorner)$.

We call σ a Γ-substitution (in some theory T), if for every atomic A, $\sigma(A) \in \Gamma$ ($\mathsf{T} \vdash \sigma(A) \leftrightarrow A'$ for some $A' \in \Gamma$). We also say that σ_T is a Γ-interpretation if σ is a Γ-substitution.

Definition 3.2. The relative provability logic of T in some sufficiently strong theory U restricted to a set of first-order sentences Γ, is defined to be a modal propositional theory $\mathcal{PL}_\Gamma(\mathsf{T}, \mathsf{U})$ such that $\mathcal{PL}_\Gamma(\mathsf{T}, \mathsf{U}) \vdash A$ iff for all arithmetical substitutions σ in Γ, we have $\mathsf{U} \vdash \sigma_\mathsf{T}(A)$. We make this convention: $\mathcal{PL}_\Gamma(\mathsf{T}, \mathbb{N})$ indicates $\mathcal{PL}_\Gamma(\mathsf{T}, \mathsf{Theory}(\mathbb{N}))$, in which $\mathsf{Theory}(\mathbb{N})$ is the set of all true sentences in the standard model of arithmetic.

Define NOI (No Outside Implication) as the set of modal propositions A, such that any occurrence of $\to$ is in the scope of some $\Box$. To be able to state an extension of Leivant's Principle (that is adequate to axiomatize Σ_1-provability logic of HA) we need a translation on the modal language which we call *Leivant's translation.* We define it recursively as follows:

- $A^l := A$ for atomic or boxed A,
- $(A \wedge B)^l := A^l \wedge B^l$,
- $(A \vee B)^l := \boxdot A^l \vee \boxdot B^l$,

- $(A \to B)^l$ is defined by cases: If $A \in \mathsf{NOI}$, we define $(A \to B)^l := A \to B^l$, otherwise we define $(A \to B)^l := A \to B$.

Let us define the box translation $(.)^\Box$ and some variants of it:

- $A^{\Box\uparrow} := A^\Box := \boxdot A$ and $A^{\Box\downarrow} := A$ for atomic A or $A := \top, \bot$,
- $(\Box A)^{\Box\uparrow} := \Box A$ and $(\Box A)^\Box := (\Box A)^{\Box\downarrow} := \Box A^\Box$,
- $(.)^{\Box\uparrow}$, $(.)^\Box$ and $(.)^{\Box\downarrow}$ commute with $\wedge$ and $\vee$,
- $(B \to C)^{\Box\uparrow} := \boxdot(B^{\Box\uparrow} \to C^{\Box\uparrow})$, $(B \to C)^\Box := \boxdot(B^\Box \to C^\Box)$ and $(B \to C)^{\Box\downarrow} := B^{\Box\downarrow} \to C^{\Box\downarrow}$.

Remark 3.3. For every A we have $A^\Box = (A^{\Box\downarrow})^{\Box\uparrow}$. Also $\mathsf{iK4} \vdash A^\Box \leftrightarrow (A^{\Box\uparrow})^{\Box\downarrow}$.

Proof. Both statements are proved easily by induction on the complexity of A, and we leave them to the reader. ❑

Definition 3.4. Let us first we list some axiom schemas:

- $\underline{\mathsf{i}} := A$, for every theorem A of $\mathsf{IPC}_\Box$,
- $\underline{\mathsf{K}} := \Box(A \to B) \to (\Box A \to \Box B)$,
- $\underline{\mathsf{4}} := \Box A \to \Box\Box A$,
- $\underline{\mathsf{Lob}} := \underline{\mathsf{L}} := \Box(\Box A \to A) \to \Box A$,
- The Completeness Principle: $\underline{\mathsf{CP}} := \underline{\mathsf{C}} := A \to \Box A$.
- Restriction of Completeness Principle to atomic variables: $\underline{\mathsf{CP_a}} := \underline{\mathsf{C_a}} := p \to \Box p$, for atomic p.
- The reflection principle: $\underline{\mathsf{S}} := \Box A \to A$.
- The complete reflection principle: $\underline{\mathsf{S}^*} := \Box A \to A^\Box$.
- The Principle of Excluded Middle: $\underline{\mathsf{PEM}} := \underline{\mathsf{P}} := A \vee \neg A$.
- Leivant's Principle: $\underline{\mathsf{Le}} := \Box(B \vee C) \to \Box(\Box B \vee C)$. [Lei75]
- Extended Leivant's Principle: $\underline{\mathsf{Le}^+} := \Box A \to \Box A^l$. [AM18]
- Trace Principle: $\underline{\mathsf{TP}} := \Box(A \to B) \to (A \vee (A \to B))$. [Vis82]
- For an axiom schema $\underline{\mathsf{A}}$, the axiom schema $\overline{\mathsf{A}}$ indicates the box of every axiom instance of $\underline{\mathsf{A}}$. Also A indicates $\underline{\mathsf{A}} \wedge \overline{\mathsf{A}}$.

All modal systems which will be defined here, only has one inference rule: modus ponens $\dfrac{B \quad B \to A}{A}$. Also the celebrated modal logics, like $\mathsf{K4}$, which has the necessitation rule of inference, $\dfrac{A}{\Box A}$, by abuse of notation, are considered here with the same name and with the same set of theorems, however without the necessitation rule. The reason for this alternate definition of systems, is quite technical. Of course one may define them with the necessitation rule, but at the cost of loosing the uniformity of definitions. So in the rest of this paper, all modal systems, are considered with the modus ponens rule of inference.

Consider a list $\mathsf{A}_1, \ldots, \mathsf{A}_n$ of axiom schemas. The notation $\mathsf{A}_1\mathsf{A}_2 \ldots \mathsf{A}_n$ will be used in this paper for a modal system containing all axiom instance of all axiom schemas A_i, and is closed under modus ponens. This genral notation makes things uniform and easy to remember for later usage. However, we make the following exceptions:

- $\mathsf{GL} := \mathsf{iGLP}$,
- $\mathsf{GL\underline{S}} := \mathsf{GL}$ plus $\underline{\mathsf{S}}$. We may define similarly $\mathsf{GL\underline{S}C_a}$ and $\mathsf{GLC_a}$.

We also gathered the list of axioms and theories in Tables 5 and 6.

Lemma 3.5. *For every modal proposition A, we have $\mathsf{iK4} \vdash A^{\Box} \leftrightarrow \boxdot A^{\Box}$.*

Proof. Use induction on the complexity of A. ❑

Lemma 3.6. *For every modal proposition A, we have $\mathsf{iK4} + \boxdot\mathsf{CP} \vdash A \leftrightarrow A^{\Box}$ and $\mathsf{iK4} + \Box\mathsf{CP} \vdash A \leftrightarrow A^{\Box\downarrow}$.*

Proof. Note that the first assertion implies the second one. To prove the equivalence of A and $A^{\Box}$ in $\mathsf{iK4} + \boxdot\mathsf{CP}$, one must use induction on the complexity of A. All cases are simple and left to the reader. ❑

Lemma 3.7. *For every modal proposition A, we have $\mathsf{iGL} \vdash A$ implies $\mathsf{iGL} \vdash A^{\Box\uparrow} \wedge A^{\Box\downarrow} \wedge A^{\Box}$. The same holds for $\mathsf{iGLC_a}$.*

Proof. Use induction on the complexity of proof $\mathsf{iGL} \vdash A$. ❑

Lemma 3.8. *Let A be some proposition and $E \in \mathsf{sub}(A^{\Box})$. Then $\mathsf{iK4} + \mathsf{CP_a} \vdash E^{\Box\uparrow} \to \boxdot E$.*

Proof. Use induction on the complexity of E. All cases are trivial except for $E = F^{\Box} \to G^{\Box}$. In this case we have $E^{\Box\uparrow} = \boxdot((F^{\Box})^{\Box\uparrow} \to (G^{\Box})^{\Box\uparrow})$. One may observe that $(A^{\Box})^{\Box\uparrow} \leftrightarrow A^{\Box}$ is valid in $\mathsf{iK4}$ and hence we have $\mathsf{iK4} \vdash E^{\Box\uparrow} \leftrightarrow \boxdot E$. ❑

3.1 Preliminaries from Arithmetic

The first-order language of arithmetic contains three functions (successor, addition and multiplication), one predicate symbol and a constant: $(\mathsf{S}, +, \cdot, \leq, 0)$. First-order intuitionistic arithmetic (HA) is the theory over IQC with the axioms:

Q1 $\mathsf{S}x \neq 0$,

Q2 $\mathsf{S}x = \mathsf{S}(y) \to x = y$,

Q3 $x + 0 = x$,

Q4 $x + \mathsf{S}y = \mathsf{S}(x + y)$,

Q5 $x.0 = 0$,

Q6 $x.\mathsf{S}y = (x.y) + x$,

Q7 $x \leq y \leftrightarrow \exists z\, z + x = y$,

Ind: For each formula $A(x)$:

$$\mathsf{Ind}(A, x) := \mathcal{UC}[(A(0) \wedge \forall x(A(x) \to A(\mathsf{S}x))) \to \forall x A(x)]$$

In which $\mathcal{UC}(B)$ is the universal closure of B.

Peano Arithmetic PA, has the same axioms of HAover CQC.

Notation 3.9. From now on, when we are working in the first-order language of arithmetic, for a first-order sentence A, the notations $\Box A$ and $\Box^{+} A$ are shorthand for $\mathsf{Prov_{HA}}(\ulcorner A \urcorner)$ and $\mathsf{Prov_{PA}}(\ulcorner A \urcorner)$, respectively. Let $i\Sigma_1$ be the theory HA, where the induction principle is restricted to Σ_1-formulae. We also define HA_x to be the theory with axioms of HA, in which the induction principle is restricted to formulae satisfying at least one of the following conditions:

- Σ_1-formulas,
- formulae with Gödel number less than x.

We define PA_x similarly. Also define $\Box_x A$ and $\Box^+_x A$ to be provability predicates in HA_x and PA_x , respectively.

Lemma 3.10. *For every formula A, we have $\mathsf{PA} \vdash \forall x\ \Box^+(\Box^+_x A \to A)$ and $\mathsf{HA} \vdash \forall x\ \Box(\Box_x A \to A)$.*

Proof. The case of PA is well known [HP93]. For the case HA, see [Smo73] or [Vis02, Theorem 8.1]. ❑

Lemma 3.11. HA *proves all true Σ_1 sentences. Moreover this argument is formalizable and provable in HA, i.e. for every Σ_1-formula $A(x_1, \ldots, x_k)$ we have $\mathsf{HA} \vdash A(x_1, \ldots, x_k) \to \Box A(\dot{x}_1, \ldots, \dot{x}_k)$.*

Proof. It is a well-known fact that any true (in the standard model $\mathbb{N}$) Σ_1-sentence is provable in HA [Vis02]. Moreover this argument is constructive and formalizable in HA. ❑

Lemma 3.12. *For any $\Delta_0(\exp)$-formula $A(\bar{x})$, we have $\mathsf{HA} \vdash \forall \bar{x}(A(\bar{x}) \vee \neg A(\bar{x}))$.*

Proof. This is well-known in the literature [TvD88]. ❑

Lemma 3.13. *Let A, B be Σ_1-formulae such that $\mathsf{PA} \vdash A \to B$. Then $\mathsf{HA} \vdash A \to B$.*

Proof. Observe that every implication of Σ_1-sentences in HA is equivalent to a Π_2 sentence and use the Π_2-conservativity of PA over HA [TvD88](3.3.4). ❑

Definition 3.14. For a first-order theory T and first-order arithmetical formula A, the Beeson-Visser translation A^{T} is defined as follows:

- $A^{\mathsf{T}} := A$ for atomic A,
- $(.)^{\mathsf{T}}$ commutes with $\wedge, \vee$ and $\exists$,
- $(A \to B)^{\mathsf{T}} := (A^{\mathsf{T}} \to B^{\mathsf{T}}) \wedge \mathsf{Prov}_{\mathsf{T}}(\ulcorner A^{\mathsf{T}} \to B^{\mathsf{T}} \urcorner)$
- $(\forall x A)^{\mathsf{T}} := \forall x A^{\mathsf{T}} \wedge \mathsf{Prov}_{\mathsf{T}}(\ulcorner \forall x A^{\mathsf{T}} \urcorner)$.

HA^* and PA^* were first introduced in [Vis82]. These theories are defined as

$$\mathsf{HA}^* := \{A \mid \mathsf{HA} \vdash A^{\mathsf{HA}}\} \quad \text{and} \quad \mathsf{PA}^* := \{A \mid \mathsf{PA} \vdash A^{\mathsf{PA}}\}.$$

Visser in [Vis82] showed that the (Σ_1-)provability logic of PA^* is iGLCT, i.e. $\mathsf{iGLCT} \vdash A$ iff for all arithmetical substitution σ, $\mathsf{PA}^* \vdash \sigma_{\mathsf{PA}^*}(A)$. That means that

$$\mathcal{PL}(\mathsf{PA}^*) = \mathcal{PL}_{\Sigma_1}(\mathsf{PA}^*) = \mathsf{iGLCT}.$$

Lemma 3.15. *For any arithmetical Σ_1-formula A*

1. $\mathsf{HA} \vdash A \leftrightarrow A^{\mathsf{HA}}$,
2. $\mathsf{HA} \vdash A \leftrightarrow A^{\mathsf{PA}}$.

Proof. See [Vis82, 4.6.iii]. ❑

Lemma 3.16. *For every arithmetical sentence A we have*

- $\mathsf{HA} \vdash \mathsf{Prov}_{\mathsf{HA}}(\ulcorner A \urcorner) \to \mathsf{Prov}_{\mathsf{HA}^*}(\ulcorner A \urcorner)$,
- $\mathsf{HA}^* \vdash A \to \mathsf{Prov}_{\mathsf{HA}}(\ulcorner A \urcorner)$,

- $\mathsf{PA}^* \vdash A \to \mathsf{Prov}_{\mathsf{PA}}(\ulcorner A \urcorner)$.

Proof. For the first item, consider some A such that $\mathsf{HA} \vdash A$. By induction on the proof of A in HA, one may prove that $\mathsf{HA} \vdash A^{\mathsf{HA}}$. Moreover this argument is formalizable and provable in HA. We refer the reader to [Vis82] for details.
For the proof of second and third items, one may use induction on the complexity of A, and we leave the routine induction to the reader. ❑

Lemma 3.17. *For any Σ_1-substitution σ and each propositional modal sentence A, we have $\mathsf{HA} \vdash (\sigma_{\mathsf{HA}^*}(A))^{\mathsf{HA}} \leftrightarrow \sigma_{\mathsf{PA}^*}(()A^{\Box\uparrow})$ and $\mathsf{PA} \vdash (\sigma_{\mathsf{PA}^*}(A))^{\mathsf{PA}} \leftrightarrow \sigma_{\mathsf{PA}^*}(A^{\Box\uparrow})$.*

Proof. Use induction on the complexity of A. All cases are easily derived by Lemma 3.15. ❑

Lemma 3.18. *For any Σ_1-substitution σ and each propositional modal sentence A, we have $\mathsf{HA} \vdash \sigma_{\mathsf{HA}}(A^{\Box}) \leftrightarrow (\sigma_{\mathsf{HA}^*}(A))^{\mathsf{HA}}$ and $\mathsf{HA} \vdash \sigma_{\mathsf{PA}}(A^{\Box}) \leftrightarrow (\sigma_{\mathsf{PA}^*}(A))^{\mathsf{PA}}$.*

Proof. Use induction on the complexity of A. All cases are easily derived by Lemma 3.15. ❑

Lemma 3.19. *For any Σ_1-substitution σ and each propositional modal sentence A, we have $\mathsf{HA} \vdash \sigma_{\mathsf{HA}}(A^{\Box\downarrow}) \leftrightarrow \sigma_{\mathsf{HA}^*}(A)$ and $\mathsf{HA} \vdash \sigma_{\mathsf{PA}}(A^{\Box\downarrow}) \leftrightarrow \sigma_{\mathsf{PA}^*}(A)$.*

Proof. We use induction on the complexity of A. All cases are easy, except for boxed case, which holds by Lemma 3.18. ❑

Lemma 3.20. *For any Σ_1-substitution σ and each propositional modal sentence A, we have $\mathsf{HA} \vdash \sigma_{\mathsf{PA}}(A^{\Box\downarrow}) \leftrightarrow \sigma_{\mathsf{PA}^*}(A)$.*

Proof. We use induction on the complexity of A. All cases are easy, except for boxed case, which holds by Lemma 3.18. ❑

3.1.1 Kripke models of HA

A first-order Kripke model for the language of arithmetic is a triple $\mathcal{K} = (K, \preccurlyeq, \mathfrak{M})$ such that:

- The frame of $\mathcal{K}$, i.e. $(K, \prec)$, is a non-empty partially ordered set,
- $\mathfrak{M}$ is a function from K to the first-order classical structures for the language of the arithmetic, i.e. $\mathfrak{M}(\alpha)$ is a first-order classical structure, for each $\alpha \in K$,
- For any $\alpha \preccurlyeq \beta \in K$, $\mathfrak{M}(\alpha)$ is a weak substructure of $\mathfrak{M}(\beta)$.

For any $\alpha \in K$ and first-order formula $A \in \mathcal{L}_\alpha$ (the language of arithmetic augmented with constant symbols $\bar{a}$ for each $a \in |\mathfrak{M}(\alpha)|$), we define $\mathcal{K}, \alpha \Vdash A$ (or simply $\alpha \Vdash A$, if no confusion is likely) inductively as follows:

- For atomic A, $\mathcal{K}, \alpha \Vdash A$ iff $\mathfrak{M}(\alpha) \models A$. Note that in the structure $\mathfrak{M}(\alpha)$, $\bar{a}$ is interpreted as a,
- $\mathcal{K}, \alpha \Vdash A \vee B$ iff $\mathcal{K}, \alpha \Vdash A$ or $\mathcal{K}, \alpha \Vdash B$,
- $\mathcal{K}, \alpha \Vdash A \wedge B$ iff $\mathcal{K}, \alpha \Vdash A$ and $\mathcal{K}, \alpha \Vdash B$,
- $\mathcal{K}, \alpha \Vdash A \to B$ iff for all $\beta \succcurlyeq \alpha$, $\mathcal{K}, \beta \Vdash A$ implies $\mathcal{K}, \beta \Vdash B$,
- $\mathcal{K}, \alpha \Vdash \exists x A$ iff $\mathcal{K}, \alpha \Vdash A[x : \bar{a}]$, for some $a \in |\mathfrak{M}(\alpha)|$,
- $\mathcal{K}, \alpha \Vdash \forall x A$ iff for all $\beta \succcurlyeq \alpha$ and $b \in |\mathfrak{M}(\beta)|$, we have $\mathcal{K}, \beta \Vdash A[x : \bar{b}]$.

It is well-known in the literature [TvD88] that HA is complete for first-order Kripke models.

Lemma 3.21. *Let $\mathcal{K} = (K, \preccurlyeq, \mathfrak{M})$ be a Kripke model of HA and A be an arbitrary Σ_1-formula. Then for each $\alpha \in K$, we have $\alpha \Vdash A$ iff $\mathfrak{M}(\alpha) \models A$.*

Proof. Use induction on the complexity of A to show that for each $\alpha \in K$, we have $\alpha \Vdash A$ iff $\mathfrak{M}(\alpha) \models A$. In the inductive step for $\to$ and $\forall$, use Lemma 3.12. ❑

3.1.2 Interpretability

Let T and S be two first-order theories. Informally speaking, we say that T interprets S ($T \rhd S$) if there exists a translation from the language of S to the language of T such that T proves the translation of all of the theorems of S. For a formal definition see [Vis98]. It is well-known that for recursive theories T and S containing PA, the assertion $T \rhd S$ is formalizable in first-order language of arithmetic. For two arithmetical sentences A and B, we use the notation $A \rhd B$ to mean that $\mathsf{PA} + A$ interprets $\mathsf{PA} + B$. The following theorem due to Orey, first appeared in [Fef60].

Theorem 3.22. *For recursive theories T and S containing* PA*, we have:*

$$\mathsf{PA} \vdash (T \rhd S) \leftrightarrow \forall x\, \Box_T \mathsf{Con}(S^x),$$

in which S^x is the restriction of the theory S to axioms with Gödel number $\leq x$ and $\mathsf{Con}(U) := \neg \Box_U \bot$.

Proof. See [Fef60]. p.80 or [Ber90]. ❑

Convention. From Theorem 3.22, one can easily observe that $\mathsf{PA} \vdash (A \rhd B) \leftrightarrow \forall x\, \Box^+(A \to \neg \Box^+_x \neg B)$. So from now on, $A \rhd B$ means its Π_2-equivalent $\forall x\, \Box^+(A \to \neg \Box^+_x \neg B)$, even when we are working in weaker theories like HA, for which the above theorem (Theorem 3.22) doesn't hold. We remind the reader that $\Box^+$ stands for provability in PA.

3.1.3 Somrýnski's method for Constructing Kripke models of HA

With the general method of constructing Kripke models for HA, invented by Smoryński [Smo73], interpretability of theories containing PA plays an important role in constructing Kripke models of HA.

Definition 3.23. A triple $\mathcal{I} := (K, \preccurlyeq, T)$ is called an I-frame iff it has the following properties:

- $(K, \preccurlyeq)$ is a finite tree,
- T is a function from K to arithmetical r.e. consistent theories containing PA,
- if $\beta \preccurlyeq \gamma$, then T_β interprets T_γ ($T_\beta \rhd T_\gamma$).

Theorem 3.24. *For every I-frame $\mathcal{I} := (K, \preccurlyeq, T)$ there exists a first-order Kripke model $\mathcal{K} = (K, \preccurlyeq, \mathfrak{M})$ such that $\mathcal{K} \Vdash$ HA and moreover $\mathfrak{M}(\alpha) \models T_\alpha$, for any $\alpha \in K$. Note that both of the I-frame and Kripke model are sharing the same frame $(K, \preccurlyeq)$.*

Proof. See [Smo73, page 372-7]. For more detailed proof of a generalization of this theorem, see [AM14, Theorem 4.8]. ❑

3.2 The NNIL formulae and related topics

The class of *No Nested Implications to the Left*, NNIL formulae in a propositional language was introduced in [VvBdJRdL95], and more explored in [Vis02]. The crucial result of [Vis02] is providing an algorithm that as input, gives a non-modal proposition A and returns its best NNIL approximation A^* from below, i.e., $\mathsf{IPC} \vdash A^* \to A$ and for all NNIL formulae B such that $\mathsf{IPC} \vdash B \to A$, we have $\mathsf{IPC} \vdash B \to A^*$. Also for all Σ_1-substitutions σ, we have $\mathsf{HA} \vdash \sigma_{\mathsf{HA}}(\Box A \leftrightarrow \Box A^*)$ [Vis02].

The precise definition of the class NNIL of modal propositions is $\mathsf{NNIL} := \{A \mid \rho A \leq 1\}$, in which the complexity measure ρ, is defined inductively as follows:

- $\rho(\Box A) = \rho(p) = \rho(\bot) = \rho(\top) = 0$, for an arbitrary atomic variables p and modal proposition A,
- $\rho(A \wedge B) = \rho(A \vee B) = \max(\rho A, \rho B)$,

- $\rho(A \to B) = \max(\rho A + 1, \rho B)$,

Definition 3.25. For any two modal propositions A and B, we define $[A]B$ by induction on the complexity of B:

- $[A]B = B$, for atomic or boxed B,
- $[A](B_1 \circ B_2) = [A](B_1) \circ [A](B_2)$ for $\circ \in \{\vee, \wedge\}$,
- $[A](B_1 \to B_2) = A' \to (B_1 \to B_2)$, in which $A' = A[B_1 \to B_2 \mid B_2]$, i.e., replace each *outer occurrence* of $B_1 \to B_2$ (by outer occurrence we mean that it is not in the scope of any $\Box$) in A by B_2,

For a set X of modal propositions, we also define $[A]X := \bigvee_{B \in X} [A]B$.

The NNIL-algorithm

For each modal proposition A, the proposition A^* is defined inductively as follows [Vis02]:

1. A is atomic or boxed, take $A^* := A$.
2. $A = B \wedge C$, take $A^* := B^* \wedge C^*$.
3. $A = B \vee C$, take $A^* := B^* \vee C^*$.
4. $A = B \to C$, we have several sub-cases. In the following, an occurrence of E in D is called an *outer occurrence*, if E is neither in the scope of an implication nor in the scope of a boxed formula.

   (a) C contains an outer occurrence of a conjunction. In this case, there is some formula $J(q)$ such that

   - q is a propositional variable not occurring in A.
   - q is outer in J and occurs exactly once.
   - $C = J[q|(D \wedge E)]$.

   Now set $C_1 := J[q|D], C_2 := J[q|E]$ and $A_1 := B \to C_1, A_2 := B \to C_2$ and finally, define $A^* := A_1^* \wedge A_2^*$.

   (b) B contains an outer occurrence of a disjunction. In this case, there is some formula $J(q)$ such that

   - q is a propositional variable not occurring in A.
   - q is outer in J and occurs exactly once.
   - $B = J[q|(D \vee E)]$.

   Now set $B_1 := J[q|D], B_2 := J[q|E]$ and $A_1 := B_1 \to C, A_2 := B_2 \to C$ and finally, define $A^* := A_1^* \wedge A_2^*$.

   (c) $B = \bigwedge X$ and $C = \bigvee Y$ and X, Y are sets of implications, atomics or boxed formulas. We have several sub-cases:

      i. X contains an atomic variable or a boxed formula E. We set $D := \bigwedge(X \setminus \{E\})$ and take $A^* := E^* \to (D \to C)^*$.

      ii. X contains $\top$. Define $D := \bigwedge(X \setminus \{\top\})$ and take $A^* := (D \to C)^*$.

      iii. X contains $\bot$. Take $A^* := \top$.

iv. X contains only implications. For any $D = E \to F \in X$, define

$$B{\downarrow}D := \bigwedge((X \setminus \{D\}) \cup \{F\}).$$

Let $Z := \{E \mid E \to F \in X\} \cup \{C\}$ and define:

$$A^* := \bigwedge\{((B{\downarrow}D) \to C)^* | D \in X\} \wedge \bigvee\{([B]E)^* \mid E \in Z\}$$

Lemma 3.26. *If* $\mathsf{IPC}_\Box \vdash A \to B$ *then* $\mathsf{IPC}_\Box \vdash A^* \to B^*$.

Proof. See [AM18, Theorem. 4.5]. ❑

The TNNIL-algorithm

Definition 3.27. TNNIL (Thoroughly NNIL) is the smallest class of propositions such that

- TNNIL contains all atomic propositions,
- if $A, B \in \mathsf{TNNIL}$, then $A \vee B, A \wedge B, \Box A \in \mathsf{TNNIL}$,
- if all $\to$ occurring in A are contained in the scope of a $\Box$ (or equivalently $A \in \mathsf{NOI}$) and $A, B \in \mathsf{TNNIL}$, then $A \to B \in \mathsf{TNNIL}$.

Let $\mathsf{TNNIL}^\Box$ indicates the set of all the propositions like $A(\Box B_1, \ldots, \Box B_n)$, such that $A(p_1, \ldots, p_n)$ is an arbitrary non-modal proposition and $B_1, \ldots, B_n \in \mathsf{TNNIL}$.

Here we define A^+ to be the TNNIL-formula approximating A. Informally speaking, to find A^+, we first compute A^* and then replace all outer boxed formula $\Box B$ in A by $\Box B^+$. More precisely, we define A^+ by induction on the maximum number of nesting $\Box$'s. Suppose that $A'(p_1, \ldots, p_n)$ and $\Box B_1, \ldots, \Box B_n$ are such that $A = A'[p_1|\Box B_1, \ldots, p_n|\Box B_n]$, where A' is a non-modal proposition and $p_1, \ldots, p_n$ are fresh atomic variables (not occurred in A). It is clear that each B_i has less number of nesting $\Box$'s and then we can define $A^+ := (A')^*[p_1|\Box B_1^+, \ldots, p_n|\Box B_n^+]$.
For a modal proposition A, let $B(p_1, \ldots, p_n)$ is the unique (modulo permutation of p_i) non-modal proposition such that $A := B(\Box C_1, \ldots, \Box C_n)$. Then define $A^- := B(\Box C_1^+, \ldots, \Box C_n^+)$. Next we may define the theory iH_σ as follows:

Definition 3.28. We define the Visser's axiom schema

$$\underline{\mathsf{V}} := A \leftrightarrow A^-$$

Then define the following modal systems:

- $\mathsf{iH}_\sigma := \mathsf{iGLLe^+V}$,
- $\mathsf{iH}_\sigma^{**} := \{\mathsf{A} : \mathsf{iH}_\sigma \vdash \mathsf{A}^\Box\}$,
- $\mathsf{iH}_\sigma^{*} := \{A : \mathsf{iH}_\sigma \vdash A^{\Box\downarrow}\}$.

Remark 3.29. The definitions of iH_σ in [AM18, sec. 4.3] and iH_σ^{**} in [AM19, def. 3.16] (which were called iH_σ^{*} there) are presented in some other equivalent way. For the sake of simplicity of definitions, we preferred Definition 3.28 here. To see an axiomatization for iH_σ^{**}, we refer the reader to [AM19].

Lemma 3.30. $\mathsf{IPC}_\Box \Vdash (A^+ \wedge (A \to B)^+) \to B^+$.

Proof. By definition of $(.)^+$, for every C we have $C^+ = (C^-)^*$. Since $\mathsf{IPC}_\Box \vdash (A^- \wedge (A \to B)^-) \to B^-$, by Lemma 3.26 we have $\mathsf{IPC}_\Box \vdash (A^- \wedge (A \to B)^-)^* \to (B^-)^*$. Then we have $\mathsf{IPC}_\Box \vdash (A^+ \wedge (A \to B)^+) \to B^+$ by the argument at the beginning of proof. ❑

Lemma 3.31. *Let A be a modal proposition. Then* $\mathsf{iK4} \vdash A^{\Box\uparrow} \leftrightarrow \boxdot A^{\Box\uparrow}$.

Proof. Use induction on the complexity of A. ❑

Lemma 3.32. *For arbitrary* $A \in \mathsf{TNNIL}^{\Box}$ *we have* $\mathsf{iK4} + \mathsf{CP_a} \vdash \boxdot A^{l} \leftrightarrow \boxdot A^{\Box\uparrow}$.

Proof. We use induction on the complexity of A:

- A is atomic: then $A^l = A$ and $A^{\Box\uparrow} = \boxdot A$. Hence by $\mathsf{CP_a}$ we have the desired equivalency.
- A is boxed: $(\Box A)^l = \Box A = (\Box A)^{\Box\uparrow}$.
- $A = B \wedge C$: then $(B \wedge C)^l = B^l \wedge C^l$ and $(B \wedge C)^{\Box\uparrow} = B^{\Box\uparrow} \wedge C^{\Box\uparrow}$. Hence by induction hypothesis we have the desired result.
- $A = B \vee C$: then $(B \vee C)^l = \boxdot B^l \vee \boxdot C^l$ and $(B \vee C)^{\Box\uparrow} = B^{\Box\uparrow} \vee C^{\Box\uparrow}$. Using Lemma 3.31 we have $\mathsf{iK4} \vdash (B \vee C)^{\Box\uparrow} \leftrightarrow (\boxdot B^{\Box\uparrow} \vee \boxdot C^{\Box\uparrow})$ and hence induction hypothesis implies the desired result.
- $A = B \to C$ and $B \in \mathsf{NOI}$: then $(B \to C)^l = B \to C^l$ and $(B \to C)^{\Box\uparrow} = \boxdot(B^{\Box\uparrow} \to C^{\Box\uparrow})$. Observe that

  1. $\mathsf{iK4} \vdash \boxdot\boxdot E \leftrightarrow \boxdot E$ for any A,
  2. $\mathsf{iK4} + \mathsf{CP_a} \vdash B^{\Box\uparrow} \leftrightarrow B$,
  3. $\mathsf{iK4} \vdash \boxdot(B \to E) \leftrightarrow \boxdot(B \to \boxdot E)$ for any $B \in \mathsf{NOI}$ and arbitrary E.

  We have the following equivalences in $\mathsf{iK4} + \mathsf{CP_a}$:

$$
\begin{array}{rcll}
\boxdot A^{\Box\uparrow} & \leftrightarrow & \boxdot(B^{\Box\uparrow} \to C^{\Box\uparrow}) & \text{by first observation}\\
\boxdot(B^{\Box\uparrow} \to C^{\Box\uparrow}) & \leftrightarrow & \boxdot(B \to C^{\Box\uparrow}) & \text{by second observation}\\
\boxdot(B \to C^{\Box\uparrow}) & \leftrightarrow & \boxdot(B \to \boxdot C^{\Box\uparrow}) & \text{by third observation}\\
\boxdot(B \to \boxdot C^{\Box\uparrow}) & \leftrightarrow & \boxdot(B \to \boxdot C^{l}) & \text{by induction hypothesis}\\
\boxdot(B \to \boxdot C^{l}) & \leftrightarrow & \boxdot(B \to C^{l}) & \text{by third observation}
\end{array}
$$

❑

Lemma 3.33. *For* $A \in \mathsf{TNNIL}^{\Box}$ *we have* $\mathsf{iK4} + \mathsf{Le}^{+} + \mathsf{CP_a} \vdash A \leftrightarrow A^{\Box\downarrow}$.

Proof. Use induction on the complexity of A. The only nontrivial case is when $A = \Box B$. We have the following equivalences in $\mathsf{iK4} + \mathsf{Le}^{+} + \mathsf{CP_a}$:

$$
\begin{array}{rcll}
(\Box B)^{\Box\downarrow} & \leftrightarrow & \Box(B^{\Box}) & \text{by definition}\\
\Box(B^{\Box}) & \leftrightarrow & \Box\left((B^{\Box\downarrow})^{\Box\uparrow}\right) & \text{by Remark 3.3}\\
\Box\left((B^{\Box\downarrow})^{\Box\uparrow}\right) & \leftrightarrow & \Box(B^{\Box\uparrow}) & \text{by induction hypothesis}\\
\Box(B^{\Box\uparrow}) & \leftrightarrow & \Box B^{l} & \text{by Lemma 3.32}\\
\Box B^{l} & \leftrightarrow & \Box B & \text{by the axiom schema } \mathsf{Le}^{+}
\end{array}
$$

❑

Theorem 3.34. *For any* TNNIL*-proposition A,* $\mathsf{iGLC} \vdash A$ *implies* $\mathsf{iGLLe}^{+} \vdash A$.

Proof. See [AM18] Theorem 4.24. ❑

Theorem 3.35. *For any* $\mathsf{TNNIL}^{\Box}$*-proposition A,* $\mathsf{iGL\overline{C}\underline{P}C_a} \vdash A$ *implies* $\mathsf{iGLLe^{+}\underline{P}} \vdash A$. *Also* $\mathsf{iGL\overline{C}\underline{SP}C_a} \vdash A$ *implies* $\mathsf{iGLLe^{+}\underline{SP}} \vdash A$.

Proof. Both statements proved by induction on proofs. The only non-trivial case is when A is an axiom instance of the form $\Box A$ such that $\mathsf{iGLC} \vdash A$. In this case, Theorem 3.34 implies $\mathsf{iGLLe}^+ \vdash A$. Hence by necessitation which is available in iGLLe^+ we have $\mathsf{iGLLe}^+ \vdash \Box A$. Hence $\mathsf{iGLLe}^+\underline{\mathsf{P}} \vdash A$ and $\mathsf{iGLLe}^+\underline{\mathsf{SP}} \vdash A$. ❑

3.3 Intuitionistic Modal Kripke Semantics

Let us first review results and notations from [Iem01] which will be used here. Assume two binary relations R and S on a set. Define $\alpha(R\,;S)\gamma$ iff there exists some β such that $\alpha R\beta$ and $\beta S\gamma$. We use the binary relation symbol $\preccurlyeq$ always as a reflexive relation and $\prec$ for the irreflexive part of $\preccurlyeq$, i.e. $u \prec v$ holds iff $u \preccurlyeq v$ and $u \neq v$. Moreover we use the mirror image of a relational symbol for its inverse, e.g. $\succ$ for $\prec^{-1}$ and so on.

A Kripke model $\mathcal{K}$, for intuitionistic modal logic, is a quadruple $(K, \preccurlyeq, \sqsubset, V)$, such that K is a set (we call its elements as nodes), $(K, \prec)$ is a partial ordering, $\sqsubset$ is a binary relation on K such that $(\preccurlyeq;\ \sqsubset) \subseteq \sqsubset$, and V is a binary relation between nodes and atomic variables such that $\alpha V p$ and $\alpha{\preccurlyeq}\beta$ implies $\beta V p$. Then we can extend V to the modal language with $\sqsubset$ corresponding to $\Box$ and $\preccurlyeq$ for intuitionistic $\to$. More precisely, we define $\Vdash$ inductively as an extension of V as follows:

- $\mathcal{K}, \alpha \Vdash p$ iff $\alpha V p$, for atomic variable p,
- $\mathcal{K}, \alpha \Vdash A \vee B$ iff $\mathcal{K}, \alpha \Vdash A$ or $\mathcal{K}, \alpha \Vdash B$,
- $\mathcal{K}, \alpha \Vdash A \wedge B$ iff $\mathcal{K}, \alpha \Vdash A$ and $\mathcal{K}, \alpha \Vdash B$,
- $\mathcal{K}, \alpha \nVdash \bot$ and $\mathcal{K}, \alpha \Vdash \top$,
- $\mathcal{K}, \alpha \Vdash A \to B$ iff for all $\beta{\succcurlyeq}\alpha$, $\mathcal{K}, \beta \Vdash A$ implies $\mathcal{K}, \beta \Vdash B$,
- $\mathcal{K}, \alpha \Vdash \Box A$ iff for all β with $\alpha \sqsubset \beta$, we have $\mathcal{K}, \beta \Vdash A$.

Also we define the local truth in this way:

- $\mathcal{K}, \alpha \models p$ iff $\alpha V p$, for atomic variable p,
- $\mathcal{K}, \alpha \models A \vee B$ iff $\mathcal{K}, \alpha \models A$ or $\mathcal{K}, \alpha \models B$,
- $\mathcal{K}, \alpha \models A \wedge B$ iff $\mathcal{K}, \alpha \models A$ and $\mathcal{K}, \alpha \models B$,
- $\mathcal{K}, \alpha \not\models \bot$ and $\mathcal{K}, \alpha \models \top$,
- $\mathcal{K}, \alpha \models A \to B$ iff either $\mathcal{K}, \alpha \not\models A$ or $\mathcal{K}, \alpha \models B$,
- $\mathcal{K}, \alpha \models \Box A$ iff for all β with $\alpha \sqsubset \beta$, we have $\mathcal{K}, \beta \Vdash A$.

The classical truth $\mathcal{K}, \alpha \models_c A$ is defined similar to $\mathcal{K}, \alpha \models A$, except for the boxed case:

- $\mathcal{K}, \alpha \models_c \Box A$ iff for all $\beta \sqsupset \alpha$ we have $\mathcal{K}, \beta \models_c A$.

For a boolean interpretation I, we also define the local I-truth $\mathcal{K}, \alpha, I \models A$ and the classical I-truth $\mathcal{K}, \alpha, I \models_c A$, similar to $\mathcal{K}, \alpha \models A$, and $\mathcal{K}, \alpha \models A$, except for atomic variables p which we define:

- $\mathcal{K}, \alpha, I \models p$ iff $I \models p$ iff $\mathcal{K}, \alpha, I \models_c p$.

Remark 3.36. Note that when we consider the classical truth for a Kripke model $\mathcal{K} = (K, \sqsubset, \preccurlyeq, V)$, we are ignoring the $\preccurlyeq$ from $\mathcal{K}$ and it would collapse to the well known Kripke semantic for the classical modal logic $\mathcal{K}_c := (K, \sqsubseteq, V)$. The same argument holds for the classical I-truth, except for the valuation V, which should be modified according to I, more precisely, $\mathcal{K}, \alpha, I \models_c A$ iff $\mathcal{K}^I_c, \alpha \models A$, in which $\mathcal{K}^I_c := (K, \sqsubset, V^I_\alpha)$ is a classical Kripke semantic for classical modal logic with

$$\beta\, V^I_\alpha\, p \quad \Leftrightarrow \quad (\beta \neq \alpha \wedge \beta\, V\, p) \vee (\beta = \alpha \wedge I \models p)$$

In the rest of paper, we may simply write $\alpha \Vdash A$ for $\mathcal{K}, \alpha \Vdash A$, if no confusion is likely. By an induction on the complexity of A, one can observe that $\alpha \Vdash A$ implies $\beta \Vdash A$ for all A and $\alpha \preccurlyeq \beta$. We define the following notions.

- If $\alpha \preccurlyeq \beta$, β is *above* α and α is *beneath* β. If $\alpha \sqsubset \beta$, β is a *successor* of α. We say that β is an immediate successor of α, if $\alpha \sqsubset \beta$ and there is no γ such that $\alpha \sqsubset \gamma \sqsubset \beta$.
- We say that α is $\sqsubset$-branching, if the set of immediate successors of α is not singleton.
- A Kripke model is finite if its set of nodes is finite.
- $(\alpha \sqsubset)$ indicates the set of successors of α, and $(\alpha \prec)$ and $(\alpha \preccurlyeq)$ are defined similarly.
- α is classical, if $(\alpha \prec) = \emptyset$.
- α is quasi-classical, if $(\alpha \prec) = (\alpha \sqsubset)$.
- α is complete if $(\alpha \sqsubset) \subseteq (\alpha \prec)$. Also we say that α is atom-complete if $\alpha \Vdash p$ and $\alpha \sqsubset \beta$ implies $\beta \Vdash p$, for every atomic variable p.
- Let φ indicates some property for nodes in $\mathcal{K}$ and $X \subseteq K$. We say that $\mathcal{K}$ is X-φ, if every $\alpha \in X$ has the property φ. If $X = \{\alpha\}$, we may use α-φ instead. We say that $\mathcal{K}$ has the property φ, or simply "is φ", if it is K-φ. For example if we set $\mathsf{Suc} := \bigcup_{\alpha \in K} (\alpha \sqsubset)$, Suc-classical means that every $\sqsubset$-accessible node is classical.
- $\mathcal{K}$ is called *neat* iff $\alpha \sqsubset \gamma$ and $\alpha \preccurlyeq \beta \preccurlyeq \gamma$ implies $\alpha \sqsubset \beta$ or $\beta \sqsubset \gamma$.
- $\mathcal{K}$ is called *brilliant* iff $(\sqsubset ; \preccurlyeq) \subseteq \sqsubset$ [Iem01]. Note that $\alpha \sqsubset ; \preccurlyeq \beta$ iff there is some δ such that $\alpha \sqsubset \delta \preccurlyeq \beta$.
- We say that $\mathcal{K}$ has tree frame, if $(K, \prec \cup \sqsubset)$ is tree. A tree is a partial order $(X, <)$ such that for every $x \in X$, the set $\{y \in X : y \leq x\}$ is finite linearly ordered.
- $\mathcal{K}$ is called *semi-perfect* iff it is (1) with finite tree frame, (2) brilliant, (3) neat and (4) $\sqsubset$ is irreflexive and transitive. We say that $\mathcal{K}$ is perfect if it is semi-perfect and complete. Note that every quasi-classical Kripke model with finite tree frame is perfect.
- We say that a Kripke model $\mathcal{K}$ is A-sound at α (α is A-sound), if for every boxed subformula $\Box B$ of A we have $\mathcal{K}, \alpha \models \Box B \to B$.
- Suppose X is a set of propositions that is closed under sub-formulae (we call such X *adequate*). An X-saturated set of propositions Γ with respect to some logic L is a consistent subset of X such that
  - For each $A \in X$, $\Gamma \vdash_L A$ implies $A \in \Gamma$.
  - For each $A \vee B \in X$, $\Gamma \vdash_L A \vee B$ implies $A \in \Gamma$ or $B \in \Gamma$.

Lemma 3.37. *Let $\nvdash_L A$ and let X be an adequate set. Then there is an X-saturated set Γ such that $\Gamma \nvdash A$.*

Proof. See [Iem01]. ❑

Theorem 3.38. iGLC *is sound and complete for perfect Kripke models. Also* iGLCT *is sound and complete for perfect quasi-classical Kripke models.*

Proof. See [AM18, Theorem 4.26] for iGLC and [Vis82, Lemma 6.14] for iGLCT. ❑

Since iGLC and iGLCT have finite model property, as it is expected, we can easily deduce the decidability of iGLC and iGLCT:

Corollary 3.39. iGLC *and* iGLCT *are decidable.*

Proof. For iGLC see [AM18, Corollary 4.27]. iGLCT is similar and left to the reader. ❑

Lemma 3.40. *Let A be a modal proposition and $\mathcal{K} = (K, \preccurlyeq, \sqsubset, V)$ be a semi-perfect Kripke model. Then for every quasi-classical node $\alpha \in K$ we have*

$$\mathcal{K}, \alpha \Vdash A^{\Box} \quad \Longleftrightarrow \quad \mathcal{K}, \alpha \models A^{\Box}$$

Proof. We use induction on the complexity of A. The only non-trivial case is when $A = B \to C$. Let $\mathcal{K}, \alpha \nVdash \boxdot(B^{\Box} \to C^{\Box})$. If $\mathcal{K}, \alpha \nVdash \Box(B^{\Box} \to C^{\Box})$ then evidently $\mathcal{K}, \alpha \not\models \Box(B^{\Box} \to C^{\Box})$ and we are done. If $\mathcal{K}, \alpha \nVdash B^{\Box} \to C^{\Box}$, then there exists some $\beta \succcurlyeq \alpha$ such that $\mathcal{K}, \beta \Vdash B^{\Box}$ and $\mathcal{K}, \beta \nVdash C^{\Box}$. Since α is quasi classical, hence $\beta \sqsupset \alpha$ or $\beta = \alpha$. If $\beta \sqsupset \alpha$, we have $\mathcal{K}, \alpha \not\models \Box(B^{\Box} \to C^{\Box})$ and we are done. Otherwise, $\mathcal{K}, \alpha \Vdash B^{\Box}$ and $\mathcal{K}, \alpha \nVdash C^{\Box}$ and hence by induction hypothesis we have $\mathcal{K}, \alpha \models B^{\Box}$ and $\mathcal{K}, \alpha \not\models C^{\Box}$ and we are done. For the other way around, let $\mathcal{K}, \alpha \not\models \boxdot(B^{\Box} \to C^{\Box})$. If $\mathcal{K}, \alpha \not\models \Box(B^{\Box} \to C^{\Box})$, evidently we have $\mathcal{K}, \alpha \nVdash \Box(B^{\Box} \to C^{\Box})$ and we are done. Otherwise, let $\mathcal{K}, \alpha \not\models B^{\Box} \to C^{\Box}$. Then $\mathcal{K}, \alpha \models B^{\Box}$ and $\mathcal{K}, \alpha \not\models C^{\Box}$. Induction hypothesis implies $\mathcal{K}, \alpha \Vdash B^{\Box}$ and $\mathcal{K}, \alpha \nVdash C^{\Box}$ and hence $\mathcal{K}, \alpha \nVdash \boxdot(B^{\Box} \to C^{\Box})$. ❑

Corollary 3.41. *Let A be a modal proposition and $\mathcal{K}$ is a semi-perfect quasi-classical Kripke model. Then for every node α we have*

$$\mathcal{K}, \alpha \Vdash A^{\Box} \quad \Longleftrightarrow \quad \mathcal{K}, \alpha \models A^{\Box} \quad \Longleftrightarrow \quad \mathcal{K}, \alpha \models_c A^{\Box}$$

$$\mathcal{K}, \alpha \models A^{\Box\downarrow} \Longleftrightarrow \mathcal{K}, \alpha \models_c A^{\Box\downarrow} \quad \textit{and} \quad \mathcal{K}, \alpha, I \models A^{\Box\downarrow} \Longleftrightarrow \mathcal{K}, \alpha, I \models_c A^{\Box\downarrow}$$

Proof. By Lemma 3.40, for every node α we have $\mathcal{K}, \alpha \Vdash A^{\Box}$ iff $\mathcal{K}, \alpha \models A^{\Box}$. One can easily observe by induction on the height of the node $\alpha \in K$ that $\mathcal{K}, \alpha \Vdash A^{\Box}$ iff $\mathcal{K}, \alpha \models_c A^{\Box}$. ❑

Corollary 3.42. *Let A be a modal proposition and $\mathcal{K}$ is a semi-perfect quasi-classical Kripke model. Then for every node α we have*

$$\mathcal{K}, \alpha \Vdash A \quad \Longleftrightarrow \quad \mathcal{K}, \alpha \models A^{\Box\uparrow}$$

Proof. Observe that $\mathcal{K}, \alpha \models B^{\Box\downarrow} \leftrightarrow B$, $\mathcal{K}, \alpha \Vdash B^{\Box} \leftrightarrow B$ and $B^{\Box} = (B^{\Box\uparrow})^{\Box\downarrow}$. Hence by Corollary 3.41 we have $\mathcal{K}, \alpha \Vdash A$ iff $\mathcal{K}, \alpha \Vdash A^{\Box}$ iff $\mathcal{K}, \alpha \models (A^{\Box\uparrow})^{\Box\downarrow}$ iff $\mathcal{K}, \alpha \models A^{\Box\uparrow}$. ❑

3.3.1 The Smorýnski Operation

In this subsection, we define the Smoryński operation on Kripke models [Smo85]. Given a Kripke model $\mathcal{K} = (K, \preccurlyeq, \sqsubset, V)$ and some fixed node $\alpha \in K$, define $\mathcal{K}' := (K', \preccurlyeq', \sqsubset', V')$ as the Kripke model constituted by adding one fresh node α' to $\mathcal{K}$. All nodes of $\mathcal{K}'$ other than α', forces the same atomic variables and have the same accessibility relationships as they did in $\mathcal{K}$. Also α' imitates all relationships of α. More precisely $\mathcal{K}'$ is constituted as follows:

- $K' := K \cup \{\alpha'\}$, in which $\alpha' \notin K$,
- $\beta \preccurlyeq' \gamma$ iff $\beta \preccurlyeq \gamma$ for every $\beta, \gamma \in K$,
- $\beta \sqsubset' \gamma$ iff $\beta \sqsubset \gamma$ for every $\beta, \gamma \in K$,
- $\beta \, V' \, p$ iff $\beta \, V \, p$ for every $\beta \in K$,
- $\alpha' \, V' \, p$ iff $\alpha \, V \, p$,
- $\alpha' \preccurlyeq' \beta$ iff ($\alpha \preccurlyeq \beta$ or $\beta = \alpha'$). Also $\beta \preccurlyeq' \alpha'$ iff $\beta = \alpha$,
- $\alpha' \sqsubset' \beta$ iff $\alpha \sqsubseteq \beta$. Also $\beta \not\sqsubset' \alpha'$ for every $\beta \in K'$.

Then we define $\mathcal{K}^{(n)}$ and α_n inductively:

- $\mathcal{K}^{(0)} := \mathcal{K}$ and $\alpha_0 := \alpha$,
- $\mathcal{K}^{(n+1)} := \left(\mathcal{K}^{(n)}\right)'$ and α_{n+1} is defined as the fresh node which is added to $\mathcal{K}^{(n)}$ in the definition of $\left(\mathcal{K}^{(n)}\right)'$.

Lemma 3.43. *Let $\mathcal{K}$ be a Kripke model which is $A^{\Box\downarrow}$-sound at the quasi-classical node α. Then for every subformula B of $A^{\Box\downarrow}$ and arbitrary boolean interpretation I we have*

1. *$\mathcal{K}, \alpha \models B$ iff $\mathcal{K}', \alpha' \models B$.*
2. *$\mathcal{K}, \alpha, I \models B$ iff $\mathcal{K}', \alpha', I \models B$.*
3. *α' is quasi-classical and $\mathcal{K}'$ is $A^{\Box\downarrow}$-sound at α'.*
4. *If $\mathcal{K}$ is semi-perfect, perfect or quasi-classical, then $\mathcal{K}'$ is so.*

Proof. 1. Use induction on the complexity of B. All cases are trivial, except for the case $B = \Box C^{\Box}$. If $\alpha' \models \Box C^{\Box}$, evidently $\alpha \models \Box C^{\Box}$ as well. If $\alpha \models \Box C^{\Box}$, then by $A^{\Box\downarrow}$-soundness, $\alpha \models C^{\Box}$, and by Lemma 3.40, $\alpha \Vdash C^{\Box}$. Hence $\alpha' \models \Box C^{\Box}$.

2. Similar to first item and left to the reader.

3. The fact that α' is quasi-classical can easily be observed by the definition of $\mathcal{K}'$ and left to the reader. The $A^{\Box\downarrow}$-soundness, is derived from first item.

4. Easy and left to the reader.

❑

4 Reduction of Arithmetical Completenesses

Let us define $[\![A; \mathsf{T}, \mathsf{U}; \Gamma]\!]$ as the set of all Γ-substitutions σ such that $\mathsf{U} \nvdash \sigma_{\mathsf{T}}(A)$. Hence $\mathcal{PL}_{\Gamma}(\mathsf{T}, \mathsf{U}) = \{A : [\![A; \mathsf{T}, \mathsf{U}; \Gamma]\!] = \emptyset\}$. For an arithmetical substitution σ, let $[\![\sigma]\!]$ indicate the propositional closure of σ, i.e. the smallest set X of arithmetical substitutions with the following conditions:

- $\sigma \in X$,
- if $\alpha \in X$, τ is some $\mathcal{L}_{\Box}$-substitution and T is some recursively axiomatizable arithmetical theory, then $\alpha_{\mathsf{T}} \circ \tau \in X$.

Note that the substitution $\alpha_{\mathsf{T}} \circ \tau$ is defined on atomic variable p in this way: $\alpha_{\mathsf{T}} \circ \tau(p) := \alpha_{\mathsf{T}}(\tau(p))$.

Let V_0 be a modal theory. We define the Γ-arithmetical completeness of V_0 with respect to T relative in U as follows:

$$\mathcal{AC}_{\Gamma}(\mathsf{V}_0; \mathsf{T}, \mathsf{U}) \equiv A \in \mathcal{PL}_{\Gamma}(\mathsf{T}, \mathsf{U}) \text{ implies } \mathsf{V}_0 \vdash A, \text{ for every } A \in \mathcal{L}_{\Box}$$

Similarly we define the Arithmetical soundness $\mathcal{AS}_{\Gamma}(\mathsf{V}_0; \mathsf{T}, \mathsf{U})$ as follows:

$$\mathcal{AS}_{\Gamma}(\mathsf{V}_0; \mathsf{T}, \mathsf{U}) \equiv \mathsf{V}_0 \vdash A \text{ implies } A \in \mathcal{PL}_{\Gamma}(\mathsf{T}, \mathsf{U}), \text{ for every } A \in \mathcal{L}_{\Box}$$

When Γ is the set of all arithmetical sentences, we may omit the subscript Γ in the notations $\mathcal{PL}_{\Gamma}(\mathsf{T}, \mathsf{U})$, $\mathcal{AC}_{\Gamma}(\mathsf{V}_0; \mathsf{T}, \mathsf{U})$ and $\mathcal{AS}_{\Gamma}(\mathsf{V}_0; \mathsf{T}, \mathsf{U})$.
Note that $\mathcal{PL}_{\Gamma}(\mathsf{T}, \mathsf{U}) = \mathsf{V}_0$ iff $\mathcal{AC}_{\Gamma}(\mathsf{V}_0; \mathsf{T}, \mathsf{U})$ and $\mathcal{AS}_{\Gamma}(\mathsf{V}_0; \mathsf{T}, \mathsf{U})$.

In the following definition, we formalize reduction of the arithmetical completeness of V_0 to V'_0:

Definition 4.1. Let T and T' be consistent recursively axiomatizable and U and U' be strong enough arithmetical theories. Also let Γ and Γ' be sets of arithmetical sentences and $\mathsf{V}_0, \mathsf{V}'_0$ be modal theories. We say that $f, \bar{f}$ propositionally reduces $\mathcal{AC}_\Gamma(\mathsf{V}_0;\mathsf{T},\mathsf{U})$ to $\mathcal{AC}_{\Gamma'}(\mathsf{V}'_0;\mathsf{T}',\mathsf{U}')$, with the notation $\mathcal{AC}_\Gamma(\mathsf{V}_0;\mathsf{T},\mathsf{U}) \leq_{f,\bar{f}} \mathcal{AC}_{\Gamma'}(\mathsf{V}'_0;\mathsf{T}',\mathsf{U}')$, if:

- R0. $f : \mathcal{L}_\Box \longrightarrow \mathcal{L}_\Box$ and $\bar{f} = \{\bar{f}_A\}_A$ is a family of functions,
- R1. $\mathsf{V}'_0 \vdash f(A)$ implies $\mathsf{V}_0 \vdash A$,
- R2. for every $A \in \mathcal{L}_\Box$, $\bar{f}_A$ is a function on arithmetical substitutions and

$$\bar{f}_A : [\![f(A);\mathsf{T}',\mathsf{U}';\Gamma']\!] \longrightarrow [\![A;\mathsf{T},\mathsf{U};\Gamma]\!] \quad \text{and for every } \alpha\text{:} \quad \bar{f}_A(\sigma) \in [\![\sigma]\!].$$

We say that $\mathcal{AC}_\Gamma(\mathsf{V}_0;\mathsf{T},\mathsf{U})$ is reducible to $\mathcal{AC}_{\Gamma'}(\mathsf{V}'_0;\mathsf{T}',\mathsf{U}')$, with the notation

$$\mathcal{AC}_\Gamma(\mathsf{V}_0;\mathsf{T},\mathsf{U}) \leq \mathcal{AC}_{\Gamma'}(\mathsf{V}'_0;\mathsf{T}',\mathsf{U}'),$$

if there exists some $f, \bar{f}$ such that $\mathcal{AC}_\Gamma(\mathsf{V}_0;\mathsf{T},\mathsf{U}) \leq_{f,\bar{f}} \mathcal{AC}_{\Gamma'}(\mathsf{V}'_0;\mathsf{T}',\mathsf{U}')$.

Following theorems are what one expect from the reduction:

Theorem 4.2. *The reduction of arithmetical completenesses is a transitive reflexive relation.*

Proof. The reflexivity is trivial and left to the reader. For the transitivity, let

$$\mathcal{AC}_\Gamma(\mathsf{V}_0;\mathsf{T},\mathsf{U}) \leq_{f,\bar{f}} \mathcal{AC}_{\Gamma'}(\mathsf{V}'_0;\mathsf{T}',\mathsf{U}') \leq_{g,\bar{g}} \mathcal{AC}_{\Gamma''}(\mathsf{V}''_0;\mathsf{T}'',\mathsf{U}'')$$

and observe that

$$\mathcal{AC}_\Gamma(\mathsf{V}_0;\mathsf{T},\mathsf{U}) \leq_{h,\bar{h}} \mathcal{AC}_{\Gamma''}(\mathsf{V}''_0;\mathsf{T}'',\mathsf{U}'')$$

in which $h := g \circ f$ and $\bar{h}_A := \bar{f}_A \circ \bar{g}_{f(A)}$. ❑

Theorem 4.3. $\mathcal{AC}_\Gamma(\mathsf{V}_0;\mathsf{T},\mathsf{U}) \leq_{f,\bar{f}} \mathcal{AC}_{\Gamma'}(\mathsf{V}'_0;\mathsf{T}',\mathsf{U}')$ *and* $\mathcal{AC}_{\Gamma'}(\mathsf{V}'_0;\mathsf{T}',\mathsf{U}')$ *implies* $\mathcal{AC}_\Gamma(\mathsf{V}_0;\mathsf{T},\mathsf{U})$.

Proof. Let $\mathsf{V}_0 \nvdash A$. Then by R1 in the Definition 4.1, $\mathsf{V}'_0 \nvdash f(A)$. Hence by $\mathcal{AC}_{\Gamma'}(\mathsf{V}'_0;\mathsf{T}',\mathsf{U}')$, there exists some Γ'-substitution σ such that $\mathsf{U}' \nvdash \sigma_{\mathsf{T}'}(f(A))$, or in other words $\sigma \in [\![f(A);\mathsf{T}',\mathsf{U}';\Gamma']\!]$. Hence by R2 $\bar{f}_A(\sigma) \in [\![A;\mathsf{T},\mathsf{U};\Gamma]\!]$, which implies $A \in \mathcal{PL}_\Gamma(\mathsf{T},\mathsf{U})$. ❑

Remark 4.4. Note that the requirement $\bar{f}_A(\alpha) \in [\![\alpha]\!]$, did not used in the proof of arithmetical completeness of V_0 in Theorem 4.3. The only usage of this condition, is to restrict the way one may compute $\bar{f}_A(\alpha)$ from α: only propositional substitutions are allowed to be composed by α to produce $\bar{f}_A(\alpha)$. If we remove this restriction from the definition, we would have a trivial reduction: every arithmetical completeness would be reducible to everyone.

Corollary 4.5. *If* $\mathcal{AC}_\Gamma(\mathsf{V}_0;\mathsf{T},\mathsf{U}) \leq_{f,\bar{f}} \mathcal{AC}_{\Gamma'}(\mathsf{V}'_0;\mathsf{T}',\mathsf{U}')$ *and* $\mathcal{AC}_{\Gamma'}(\mathsf{V}'_0;\mathsf{T}',\mathsf{U}')$, *then we have*

$$\mathsf{V}_0 \vdash A \quad \Longleftrightarrow \quad \mathsf{V}'_0 \vdash f(A)$$

Proof. The direction $\Longleftarrow$ holds by definition. For the other way around, use Theorem 4.3. ❑

Remark 4.6. Note that $\mathsf{V}_0 \vdash A \Longleftrightarrow \mathsf{V}'_0 \vdash f(A)$ is not enough for reduction of the arithmetical completenesses. This is simply because f does not have anything to do with the arithmetical substitutions. So one may not be able to translate an arithmetical refutation from $\mathcal{PL}_{\Gamma'}(\mathsf{T}',\mathsf{U}')$ to a refutation from $\mathcal{PL}_\Gamma(\mathsf{T},\mathsf{U})$, via propositional translations. If we remove the condition 3 and replace second item by $\mathsf{V}_0 \vdash A \Longleftrightarrow \mathsf{V}'_0 \vdash f(A)$ in the Definition 4.1, $\mathcal{AC}_\Gamma(\mathsf{V}_0;\mathsf{T},\mathsf{U})$ would be reducible to every arithmetical completeness via the following vicious reduction:

$$f(A) := \begin{cases} \top & : \text{ if } \mathsf{V}_0 \vdash A \\ \bot & : \text{ otherwise} \end{cases}$$

Notation 4.7. In the rest of the paper, we are going to characterize several provability logics. Our main tool for proving their arithmetical completeness is the reduction of arithmetical completenesses Theorem 4.3. The notation $\mathsf{V}_0 = \mathcal{PL}_\Gamma(\mathsf{T},\mathsf{U}) \leq \mathcal{PL}_{\Gamma'}(\mathsf{T}',\mathsf{U}') = \mathsf{V}_0'$ means that the following items hold: (1) $\mathcal{AC}_\Gamma(\mathsf{V}_0;\mathsf{T},\mathsf{U}) \leq \mathcal{AC}_{\Gamma'}(\mathsf{V}_0';\mathsf{T}',\mathsf{U}')$, (2) $\mathsf{V}_0 = \mathcal{PL}_\Gamma(\mathsf{T},\mathsf{U})$, (3) $\mathcal{PL}_{\Gamma'}(\mathsf{T}',\mathsf{U}') = \mathsf{V}_0'$.
Let the provability logics $\mathcal{PL}_\Gamma(\mathsf{T},\mathsf{U}) = \mathsf{V}_0$ and $\mathcal{PL}_{\Gamma'}(\mathsf{T}',\mathsf{U}') = \mathsf{V}_0'$ are already characterized. Then the notation $\mathcal{PL}_\Gamma(\mathsf{T},\mathsf{U}) \leq \mathcal{PL}_{\Gamma'}(\mathsf{T}',\mathsf{U}')$ indicates $\mathcal{AC}_\Gamma(\mathsf{V}_0;\mathsf{T},\mathsf{U}) \leq \mathcal{AC}_{\Gamma'}(\mathsf{V}_0';\mathsf{T}',\mathsf{U}')$.

Theorem 4.8. *Let $\mathcal{PL}_\Gamma(\mathsf{T},\mathsf{U}) \leq_{f,\bar{f}} \mathcal{PL}_{\Gamma'}(\mathsf{T}',\mathsf{U}')$ for some computable function f. Then the decidability of $\mathcal{PL}_{\Gamma'}(\mathsf{T}',\mathsf{U}')$ implies the decidability of $\mathcal{PL}_\Gamma(\mathsf{T},\mathsf{U})$.*

Proof. Direct consequence of Corollary 4.5 and computability of f. ❑

4.1 Two special cases

In later applications, always we consider two simple cases of reduction $f, \bar{f}$ (Definition 4.1) to provide new arithmetical completenesses:

- Identity: in this case we consider $\bar{f}_A$ as identity function and $f(A)$ is some propositional translation like $(.)^\Box$ or $(.)^{\Box\downarrow}$.
- Substitution: in this case, we let $f(A)$ as some $\mathcal{L}_\Box$-substitution, possibly depending on A. Also $\bar{f}_A(\sigma) := \sigma_{\mathsf{T}'} \circ \tau$.

5 Relative Σ_1-provability logics for HA

In this section, we will characterize $\mathcal{PL}_{\Sigma_1}(\mathsf{HA},\mathbb{N})$, i.e. the truth Σ_1-provability logic of HA, and $\mathcal{PL}_{\Sigma_1}(\mathsf{HA},\mathsf{PA})$, i.e. the Σ_1-provability logic of HA, relative to PA. We also show that $\mathcal{PL}_{\Sigma_1}(\mathsf{HA},\mathbb{N})$ is hardest among the Σ_1-provability logics of HA relative in HA, PA, $\mathbb{N}$. In other words:

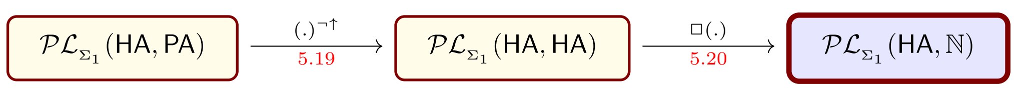

Diagram 1: Reductions for relative provability logics of HA

5.1 Kripke Semantic

Lemma 5.1. *For every A we have*

$$\mathsf{iGLC} \vdash A \quad \Longleftrightarrow \quad \mathsf{iGL} \vdash \left[\boxdot \bigwedge_{E \in \mathsf{sub}(A)} (E \to \Box E) \right] \to A$$

Proof. For the simplicity of notations, in this proof, let

$$\varphi := \boxdot \bigwedge_{E \in \mathsf{sub}(A)} (E \to \Box E)$$

and $\vdash$ indicates derivablity in $\mathsf{iGL} + \varphi$.
One side is trivial. For the other way around, assume that $\mathsf{iGL} \nvdash \varphi \to A$. We will construct some perfect Kripke model $\mathcal{K} = (K, \sqsubset, \preccurlyeq, V)$ such that $\mathcal{K}, \alpha \nVdash A$, which by soundness of iGLC for finite brilliant models with $\sqsubset \subseteq \prec$, we have the desired result. The proof is almost identical to the proof of Theorem 3.38 in [AM18, Theorem 4.26], but to be self-contained, we repeat it here.

Let $\mathsf{Sub}(A)$ be the set of sub-formulae of A. Then define

$$X := \{B, \Box B \mid B \in \mathsf{Sub}(A)\}$$

It is obvious that X is a finite adequate set. We define $\mathcal{K} = (K, \preccurlyeq, \sqsubset, V)$ as follows. Take K as the set of all X-saturated sets with respect to $\mathsf{iGL} + \varphi$, and $\preccurlyeq$ is the subset relation over K. Define $\alpha \sqsubset \beta$ iff for all $\Box B \in X$, $\Box B \in \alpha$ implies $B \in \beta$, and also there exists some $\Box C \in \beta \setminus \alpha$. Finally define $\alpha V p$ iff $p \in \alpha$, for atomic p.
It only remains to show that $\mathcal{K}$ is a finite brilliant Kripke model with $\sqsubset \subseteq \prec$ which refutes A. To this end, we first show by induction on $B \in X$ that $B \in \alpha$ iff $\alpha \Vdash B$, for each $\alpha \in K$. The only non-trivial case is $B = \Box C$. Let $\Box C \notin \alpha$. We must show $\alpha \nVdash \Box C$. The other direction is easier to prove and we leave it to reader. Let $\beta_0 := \{D \in X \mid \alpha \vdash \Box D\}$. If $\beta_0, \Box C \vdash C$, since by definition of β_0, we have $\alpha \vdash \Box \beta_0$ and hence by Löb's axiom, $\alpha \vdash \Box C$, which is in contradiction with $\Box C \notin \alpha$. Hence $\beta_0, \Box C \nvdash C$ and so there exists some X-saturated set β such that $\beta \nvdash C$, $\beta \supseteq \beta_0 \cup \{\Box C\}$. Hence $\beta \in K$ and $\alpha \sqsubset \beta$. Then by the induction hypothesis, $\beta \nVdash C$ and hence $\alpha \nVdash \Box C$.

Since $\mathsf{iGL} + \varphi \nvdash A$, by Lemma 3.37, there exists some X-saturated set $\alpha \in K$ such that $\alpha \nvdash A$, and hence by the above argument we have $\alpha \nVdash A$.

$\mathcal{K}$ trivially satisfies all the properties of finite brilliant Kripke model with $\sqsubset \subseteq \prec$. As a sample, we show that why $\sqsubset \subseteq \prec$ holds. Assume $\alpha \sqsubset \beta$ and let $B \in \alpha$. If $B = \Box C$ for some C, then by definition, $C \in \beta$ and since $C \to \Box C$ is a conjunct in φ, we have $\beta \vdash \Box C$ and we are done. So assume B is not a boxed formula. Then by definition of X, we have $\Box B \in X$ and since $B \to \Box B$ is a conjunct in φ, we have $\alpha \vdash \Box B$ and hence by definition of $\sqsubset$, it is the case that $B \in \beta$. This shows $\alpha \subseteq \beta$ and hence $\alpha \preccurlyeq \beta$. But α is not equal to β, because $\alpha \sqsubset \beta$ implies existence of some $\Box C \in \beta \setminus \alpha$. Hence $\alpha \prec \beta$, as desired. ❑

Lemma 5.2. *For any proposition A, if $\mathsf{iGLC} \vdash A^\Box$ then $\mathsf{iGL} + \mathsf{CP_a} + \Box\mathsf{CP} \vdash A^\Box$.*

Proof. Let $\mathsf{iGLC} \vdash A$. Hence by Lemma 5.1 for some finite set X of subformulas of $A^\Box$ we have

$$\mathsf{iGL} \vdash \boxdot \left(\bigwedge_{E \in X} E \to \Box E \right) \to A^\Box$$

Lemma 3.7 implies

$$\mathsf{iGL} \vdash \Box \left(\bigwedge_{E \in X} E \to \Box E \right) \wedge \left(\bigwedge_{E \in X} \boxdot (E^{\Box\uparrow} \to \Box E) \right) \to A^\Box$$

By Lemma 3.8 we have $\mathsf{iGL} + \mathsf{CP_a} + \Box\mathsf{CP} \vdash A^\Box$. ❑

Theorem 5.3. $\mathsf{iGL\overline{C}\underline{P}C_a}$ *is sound and complete for local truth at quasi-classical nodes in perfect Kripke models. More precisely, we have $\mathsf{iGL\overline{C}\underline{P}C_a} \vdash A$ iff $\mathcal{K}, \alpha \models A$ for every perfect Kripke model $\mathcal{K}$ and the quasi-classical node α.*

Proof. The soundness part easily derived by the soundness of iGLC and left to the reader.
Since local truth at α is not affected by changing the set of $\preccurlyeq$-accessible nodes from α, it is enough to prove the completeness part only for the perfect Kripke models. Let $\mathsf{iGL\overline{C}\underline{P}C_a} \nvdash A$. Let A' be a boolean equivalent of A which is a conjunction of implications $E \to F$ in which E is a conjunction of a set of atomics or boxed propositions and F is a disjunction of atomics or boxed proposition. Evidently such A' exists for every A. Hence $\mathsf{iGL\overline{C}\underline{P}C_a} \nvdash A'$. Then there must be some conjunct $E \to F$ of A' such that $\mathsf{iGL\overline{C}\underline{P}C_a} \nvdash (E \to F)^\Box$, E is a conjunction of atomic and boxed propositions and F is a disjunction of atomic and boxed propositions. Hence $\mathsf{iGL} + \mathsf{CP_a} + \Box\mathsf{CP} \nvdash (E \to F)^\Box$ and by Lemma 5.2 we have $\mathsf{iGLC} \nvdash (E \to F)^\Box$. By Theorem 3.38, there exists some perfect Kripke model $\mathcal{K} = (K, \preccurlyeq, \sqsubset, V)$ such that $\mathcal{K}, \alpha \nVdash (E \to F)^\Box$ for some $\alpha \in K$. Since iGLC is sound for $\mathcal{K}$,

we have $\mathcal{K},\alpha \not\Vdash E \to F$. Hence there exists some $\beta \succcurlyeq \alpha$ such that $\mathcal{K},\beta \Vdash E$ and $\mathcal{K},\beta \not\Vdash F$. Then by definition of local truth we have $\mathcal{K},\beta \models E$ and $\mathcal{K},\beta \not\models F$. Then $\mathcal{K},\beta \not\models E \to F$. Hence $\mathcal{K},\beta \not\models A$, as desired. ❑

Corollary 5.4. $\mathsf{iGL}\overline{\mathsf{C}}\underline{\mathsf{P}}\mathsf{C_a}$ *is decidable.*

Proof. Direct consequence of the proof of Theorem 5.3 and decidability of iGLC (Corollary 3.39). ❑

Theorem 5.5. $\mathsf{iGL}\overline{\mathsf{C}}\underline{\mathsf{SP}}\mathsf{C_a} \vdash A^{\Box\downarrow}$ *iff* $\mathcal{K},\alpha \models A^{\Box\downarrow}$ *for every perfect Kripke models* $\mathcal{K}$ *and quasi-classical* $A^{\Box\downarrow}$*-sound nodes* α.

Proof. Both directions are non-trivial and proved contra-positively. For the soundness part, assume that $\mathcal{K},\alpha \not\models A^{\Box\downarrow}$ for some perfect Kripke model $\mathcal{K} := (K, \preccurlyeq, \sqsubset, V)$ which is $A^{\Box\downarrow}$-sound at the quasi-classical node $\alpha \in K$. Since derivability is finite, it is enough to show that for every finite set Γ of modal propositions we have

$$\mathsf{iGL}\overline{\mathsf{C}}\underline{\mathsf{P}}\mathsf{C_a} \nvdash \bigwedge_{B\in\Gamma}(\Box B \to B) \to A^{\Box\downarrow}.$$

By Theorem 5.3 and Lemma 3.43, it is enough to find some number i such that

$$\mathcal{K}^{(i)}, \alpha_i \not\models \bigwedge_{B\in\Gamma}(\Box B \to B) \to A^{\Box\downarrow}.$$

Let us define n_i and m_i as the number of propositions in the sets $N_i := \{B \in \Gamma : \mathcal{K}^{(i)}, \alpha_i \models B \wedge \Box B\}$ and $M_i := \{B \in \Gamma : \mathcal{K}^{(i)}, \alpha_i \models \Box B \wedge \neg B\}$, respectively.
We use induction as follows. As induction hypothesis, assume that for any number i with $n_i < k$ there is some $0 \leq j \leq 1 + n_i$ such that

$$\mathcal{K}^{(i+j)}, \alpha_{i+j} \not\models \bigwedge_{B\in\Gamma}(\Box B \to B) \to A^{\Box\downarrow} \tag{5.1}$$

Let $n_i = k$. If $m_i = 0$, we may let $j = 0$ and by Lemma 3.43 we have eq. (5.1) as desired. So let $B \in \Gamma$ such that $\mathcal{K}^{(i)}, \alpha_i \models \Box B \wedge \neg B$. We have two sub-cases:

- $m_{i+1} = 0$: observe in this case that eq. (5.1) holds for $j = 1$.
- $m_{i+1} > 0$: in this case we have $n_{i+1} < k$ and hence by application of the induction hypothesis with $i := i + 1$, we get some $0 \leq j' \leq 1 + n_{i+1}$ such that

  $$\mathcal{K}^{(i+1+j')}, \alpha_{i+1+j'} \not\models \bigwedge_{B\in\Gamma}(\Box B \to B) \to A^{\Box\downarrow}$$

  Hence if we let $j := j' + 1$ we have $0 \leq j \leq 1 + n_i$ and eq. (5.1), as desired.

For the completeness part, assume that $\mathsf{iGL}\overline{\mathsf{C}}\underline{\mathsf{SP}}\mathsf{C_a} \nvdash A^{\Box\downarrow}$. Hence

$$\mathsf{iGL}\overline{\mathsf{C}}\underline{\mathsf{P}}\mathsf{C_a} \nvdash \left(\bigwedge_{\Box B\in\mathsf{Sub}(A)}(\Box B \to B)\right) \to A^{\Box\downarrow}$$

Hence Theorem 5.3 implies the desired result. ❑

Corollary 5.6. $\mathsf{iGL}\overline{\mathsf{C}}\underline{\mathsf{SP}}\mathsf{C_a}$ *is decidable.*

Proof. First observe that by Theorems 5.3 and 5.5, we have $\mathsf{iGL}\overline{\mathsf{C}}\underline{\mathsf{SP}}\mathsf{C_a} \vdash A^{\Box\downarrow}$ iff

$$\mathsf{iGL}\overline{\mathsf{C}}\underline{\mathsf{P}}\mathsf{C_a} \vdash \bigwedge_{\Box B\in\mathsf{Sub}(A^{\Box\downarrow})}(\Box B \to B) \to A^{\Box\downarrow}.$$

Hence the decidability of $\mathsf{iGL}\overline{\mathsf{C}}\underline{\mathsf{P}}\mathsf{C_a}$ (Corollary 5.4) implies the decidability of $\mathsf{iGL}\overline{\mathsf{C}}\underline{\mathsf{SP}}\mathsf{C_a}$. ❑

5.2 Arithmetical interpretations

The following theorem is the main result in [AM18]:

Theorem 5.7. iH_σ *is the* Σ_1*-provability logic of* HA*, i.e.* $\mathsf{iH}_\sigma \vdash A$ *iff for all* Σ_1*-substitution* σ *we have* $\mathsf{HA} \vdash \sigma_{\mathsf{HA}}(A)$*. Moreover* iH_σ *is decidable.*

Here we bring some essential facts and definitions from [AM18]. Let us fix some perfect Kripke model $\mathcal{K}_0 = (K_0, \sqsubset_0, \preccurlyeq_0, V_0)$ with the quasi-classical root α_0 and its extension $\mathcal{K} := \mathcal{K}_0' = (K, \preccurlyeq, \sqsubset, V)$ by the Smoryński operation with the new quasi-classical root α_1 (which was called α_0 in [AM18]) and define a recursive function F, called Solovay function, as we did in [AM18]. We have the following definitions and facts from [AM18]: (later we refer to them simply as e.g. "item 1",)

1. The function F is provably total in HA and hence we may use the function symbol F inside HA and stronger theories.
2. The Σ_1-substitution σ is defined in this way:
$$\sigma(p) := \bigvee_{\mathcal{K},\alpha \Vdash p} \exists x (F(x) = \alpha)$$
3. Define $L = \alpha$ as $\exists x \forall y \geq x F(y) = \alpha$.
4. $\mathsf{PA} \vdash \exists x\, (F(x) = \alpha) \to \bigvee_{\beta \succcurlyeq \alpha} L = \beta$. [AM18, Lemma 5.2]
5. For a modal proposition A when we use A in a context which it is expected to be some first-order formula, like $\mathsf{HA} \vdash A$, we should replace A with the first-order sentence $\sigma_{\mathsf{HA}}(A)$.
6. For every $A \in \mathsf{sub}(\Gamma) \cap \mathsf{TNNIL}$ and $\alpha \in K_0$ such that $\mathcal{K}_0, \alpha \Vdash A$, we have $\mathsf{HA} \vdash \exists x\, F(x) = \alpha \to A$. [AM18, Lemma 5.18 & 5.19]
7. For each $B \in \mathsf{Sub}(\Gamma) \cap \mathsf{TNNIL}$ and $\alpha \in K$ such that $\alpha \nVdash \Box B$,
$$\mathsf{HA} \vdash L{=}\alpha \to \neg \Box B.$$
8. $\mathbb{N} \models L = \alpha_1$ and $\mathsf{PA} + L = \alpha$ is consistent for every $\alpha \in K$. [AM18, Corollaries 5.20 & 5.24 and Lemma 5.23]

Lemma 5.8. *For every* $A \in \mathsf{NOI} \cap \mathsf{sub}(\Gamma)$ *such that* $\mathcal{K}, \alpha_1 \nVdash A$*, we have*

$$\mathsf{HA} \vdash A \leftrightarrow \bigvee_{\alpha \in K \text{ and } \mathcal{K},\alpha \Vdash A} \exists x\, F(x) = \alpha$$

Proof. First observe that by Π_2-conservativity of PA over HA (Lemma 3.13), it is enough to prove this lemma in PA instead of HA. Then by "item 4", it is enough to show that

$$\mathsf{PA} \vdash A \leftrightarrow \bigvee_{\alpha \in K \text{ and } \mathcal{K},\alpha \Vdash A} L = \alpha$$

We use induction on the complexity of A. Since $A \in \mathsf{NOI}$ we do not consider the $\to$ case in the induction steps:

- A is atomic: by definition of the arithmetical substitution, we have
$$\sigma(A) = \bigvee_{\alpha \in K \text{ and } \mathcal{K},\alpha \Vdash A} \exists x\, F(x) = \alpha.$$

- $A = B \circ C$ and $\circ \in \{\vee, \wedge\}$: easy and left to the reader.
- $A = \Box B$: first note that by "item 6", $\mathsf{PA} \vdash \exists x\, F(x) = \alpha \to A$ for every $\alpha \Vdash A$ (here actually we need $\alpha_1 \nVdash A$). Hence

$$\mathsf{PA} \vdash \bigvee_{\alpha \in K \text{ and } \mathcal{K}, \alpha \Vdash A} \exists x\, F(x) = \alpha \to A$$

For the other direction, it is enough (by "item 4") to show that for every $\beta \in K$ such that $\mathcal{K}, \beta \nVdash A$ we have $\mathsf{PA} \vdash A \to L \neq \beta$ or equivalently $\mathsf{PA} \vdash L = \beta \to \neg A$, which holds by "item 7". ❑

Lemma 5.9. *For every $A \in \mathsf{sub}(\Gamma)$ and $\alpha \in K_0$, we have*

$$\begin{cases} \mathcal{K}, \alpha \models A & \Longrightarrow \quad \mathsf{HA} \vdash L = \alpha \to A \\ \mathcal{K}, \alpha \not\models A & \Longrightarrow \quad \mathsf{HA} \vdash L = \alpha \to \neg A \end{cases}$$

Proof. We use induction on the complexity of A. All cases are simple and we only treat the case $A = \Box B$ here. If $\mathcal{K}, \alpha \models \Box B$, by definition, $\mathcal{K}, \alpha \Vdash \Box B$ and hence by "item 6" we have the desired result. If also $\mathcal{K}, \alpha \not\models \Box B$, by definition, $\mathcal{K}, \alpha \nVdash \Box B$ and hence by "item 7" we have the desired result. ❑

Lemma 5.10. *Let $\mathcal{K}$ be $A^{\Box\downarrow}$-sound at α_0 for some $A^{\Box\downarrow} \in \mathsf{sub}(\Gamma)$. Then for every $B \in \mathsf{sub}(A^{\Box\downarrow})$ we have*

$$\begin{cases} \mathcal{K}, \alpha_1 \models B & \Longrightarrow \quad \mathsf{HA} \vdash L = \alpha_1 \to B \\ \mathcal{K}, \alpha_1 \not\models B & \Longrightarrow \quad \mathsf{HA} \vdash L = \alpha_1 \to \neg B \\ \mathcal{K}, \alpha_1 \Vdash B & \Longrightarrow \quad \mathsf{HA} \vdash B \end{cases}$$

Proof. We prove this by induction on the complexity of $B \in \mathsf{sub}(A^{\Box\downarrow})$.

- B is atomic, conjunction or disjunction: easy and left to the reader.
- $B = E \to F$: it is easy to show the first two derivations and we leave them to the reader. For the third one, assume that $\mathcal{K}, \alpha_1 \Vdash E \to F$. If $\mathcal{K}, \alpha_1 \Vdash F$ we have the desired result by induction hypothesis. So let $\mathcal{K}, \alpha_1 \nVdash F$ and Hence $\mathcal{K}, \alpha_1 \nVdash E$. Hence by Lemma 5.8, we have $\mathsf{HA} \vdash E \to \bigvee_{\alpha \Vdash E} \exists x\, F(x) = \alpha$. On the other hand by "item 6" we have $\mathsf{HA} \vdash \bigvee_{\alpha \Vdash E} \exists x\, F(x) = \alpha \to F$. Hence we have $\mathsf{HA} \to E \to F$.
- $B = \Box C^{\Box}$: Let $\mathcal{K}, \alpha_1 \models \Box C^{\Box}$. Then by Lemma 3.43 we have $\mathcal{K}, \alpha_1 \models C^{\Box}$ and hence by Lemma 3.40 $\mathcal{K}, \alpha_1 \Vdash C^{\Box}$. Then by induction hypothesis $\mathsf{HA} \vdash C^{\Box}$ and hence $\mathsf{HA} \vdash L = \alpha_1 \to \Box C^{\Box}$.
  For the second derivation, Let $\mathcal{K}, \alpha_1 \not\models \Box C^{\Box}$. Then by "item 7" we have the desired result.
  For the third derivation, let $\mathcal{K}, \alpha_1 \Vdash \Box C^{\Box}$. Then by Lemma 3.40 we have $\mathcal{K}, \alpha_1 \models \Box C^{\Box}$ and hence Lemma 3.43 implies $\mathcal{K}, \alpha_1 \models C^{\Box}$ and then again byLemma 3.40 $\mathcal{K}, \alpha_1 \Vdash C^{\Box}$. Then by induction hypothesis $\mathsf{HA} \vdash C^{\Box}$ and hence $\mathsf{HA} \vdash \Box C^{\Box}$.

❑

5.3 Arithmetical Completeness

Definition 5.11. Define the following modal systems:

- $\mathsf{iH}_\sigma\underline{\mathsf{P}} := \mathsf{iH}_\sigma$ plus $\underline{\mathsf{P}}$,
- $\mathsf{iH}_\sigma\underline{\mathsf{SP}} := \mathsf{iH}_\sigma\underline{\mathsf{P}}$ plus $\underline{\mathsf{S}}$,

- $\mathsf{iH}_\sigma\underline{\mathsf{P}}^* := \{A \in \mathcal{L}_\Box : \mathsf{iH}_\sigma\underline{\mathsf{P}} \vdash A^{\Box\downarrow}\}$,
- $\mathsf{iH}_\sigma\underline{\mathsf{SP}}^* := \{A \in \mathcal{L}_\Box : \mathsf{iH}_\sigma\underline{\mathsf{SP}} \vdash A^{\Box\downarrow}\}$.

Obviously $\mathsf{iH}_\sigma\underline{\mathsf{SP}}^*$ and $\mathsf{iH}_\sigma\underline{\mathsf{P}}^*$ are closed under modus ponens.

Theorem 5.12. $\mathsf{iH}_\sigma\underline{\mathsf{P}} = \mathcal{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{PA})$, *i.e.* $\mathsf{iH}_\sigma\underline{\mathsf{P}}$ *is the relative* Σ_1*-provability logic of* HA *in* PA.

Proof. The soundness easily deduced by use of the soundness of the iH_σ for arithmetical interpretations in HA (see Theorem 6.3 in [AM18]).

For the other way around, let $\mathsf{iH}_\sigma\underline{\mathsf{P}} \nvdash A$. Then $\mathsf{iH}_\sigma\underline{\mathsf{P}} \nvdash A^-$ in which $A^- \in \mathsf{TNNIL}^\Box$ and $\mathsf{iH}_\sigma \vdash A \leftrightarrow A^-$. Then $\mathsf{iGLLe}^+\underline{\mathsf{P}} \nvdash A^-$ and hence by Theorem 3.35 we have $\mathsf{iGL}\overline{\mathsf{C}}\underline{\mathsf{P}}\mathsf{C_a} \nvdash A^-$. By Theorem 5.3, there is some perfect Kripke model $\mathcal{K}_0$ with the quasi-classical root α_0 such that $\mathcal{K}_0, \alpha_0 \not\models A^-$. Let σ be the Σ_1-substitution as provided in Section 5.2 for the Kripke model $\mathcal{K}_0$ and its Smoryński extension $\mathcal{K}$ with $\Gamma := \{A^-\}$. Then by Lemma 5.9 we have $\mathsf{HA} \vdash L = \alpha_0 \to \sigma_{\mathsf{HA}}(\neg A^-)$. Since $\mathsf{iH}_\sigma \vdash A \leftrightarrow A^-$, by soundness part of Theorem 5.7 we have $\mathsf{HA} \vdash L = \alpha_0 \to \sigma_{\mathsf{HA}}(\neg A)$. Hence by "item 8" we may deduce $\mathsf{PA} \nvdash \sigma_{\mathsf{HA}}(A)$, as desired. ❑

Theorem 5.13. $\mathsf{iH}_\sigma\underline{\mathsf{SP}} = \mathcal{PL}_{\Sigma_1}(\mathsf{HA}, \mathbb{N})$, *i.e.* $\mathsf{iH}_\sigma\underline{\mathsf{SP}}$ *is the truth* Σ_1*-provability logic of* HA.

Proof. The soundness easily deduced by use of the soundness of the iH_σ for arithmetical interpretations in HA (see Theorem 6.3 in [AM18]).

For the other way around, let $\mathsf{iH}_\sigma\underline{\mathsf{SP}} \nvdash A$. Then by Lemma 3.33 we have $\mathsf{iH}_\sigma\underline{\mathsf{SP}} \nvdash (A^-)^{\Box\downarrow}$ in which $(A^-)^{\Box\downarrow} \in \mathsf{TNNIL}^\Box$ and $\mathsf{iH}_\sigma \vdash A \leftrightarrow (A^-)^{\Box\downarrow}$. Then $\mathsf{iGLLe}^+\underline{\mathsf{SP}} \nvdash (A^-)^{\Box\downarrow}$ and hence by Theorem 3.35 we have $\mathsf{iGL}\overline{\mathsf{C}}\underline{\mathsf{SP}}\mathsf{C_a} \nvdash (A^-)^{\Box\downarrow}$.

By Theorem 5.5, there is some perfect Kripke model $\mathcal{K}_0$ with the quasi-classical $(A^-)^{\Box\downarrow}$-sound root α_0 such that $\mathcal{K}_0, \alpha_0 \not\models (A^-)^{\Box\downarrow}$. Let σ be the Σ_1-substitution as provided in Section 5.2 for the Kripke model $\mathcal{K}_0$ and its Smoryński extension $\mathcal{K}$ with $\Gamma := \{(A^-)^{\Box\downarrow}\}$. Then by Lemma 5.10 we have $\mathsf{HA} \vdash L = \alpha_1 \to \sigma_{\mathsf{HA}}(\neg(A^-)^{\Box\downarrow})$. Since $\mathsf{iH}_\sigma \vdash A \leftrightarrow (A^-)^{\Box\downarrow}$, by soundness part of the Theorem 5.7 we have $\mathsf{HA} \vdash L = \alpha_1 \to \sigma_{\mathsf{HA}}(\neg A)$. Hence by "item 8" we may deduce $\mathbb{N} \not\models \sigma_{\mathsf{HA}}(A)$, as desired. ❑

5.4 Reductions

In this subsection we will show that

$$\mathcal{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{PA}) \le \mathcal{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{HA}) \le \mathcal{PL}_{\Sigma_1}(\mathsf{HA}, \mathbb{N})$$

First some definition:

Definition 5.14. For $A \in \mathcal{L}_\Box$ we define $A^{\neg\downarrow}, A^{\neg\uparrow}$ and $A^\neg$ as follows:

- $(A \circ B)^\neg := \neg\neg(A^\neg \circ B^\neg)$, $(A \circ B)^{\neg\uparrow} := \neg\neg(A^{\neg\uparrow} \circ B^{\neg\uparrow})$ and $(A \circ B)^{\neg\downarrow} := A^{\neg\downarrow} \circ B^{\neg\downarrow}$,
- $(\Box A)^\neg := \neg\neg\Box A^\neg$, $(\Box A)^{\neg\uparrow} := \neg\neg\Box A$ and $(\Box A)^{\neg\downarrow} := \Box A^\neg$,
- $(\neg A)^\neg := \neg A^\neg$, $(\neg A)^{\neg\uparrow} := \neg A^{\neg\uparrow}$ and $(\neg A)^{\neg\downarrow} := \neg A^{\neg\downarrow}$,
- $p^\neg := p^{\neg\uparrow} := \neg\neg p$ and $p^{\neg\downarrow} := p$ for atomic p.

For an arithmetical formula A we have these additional clauses for the definition of $A^\neg$:

- $(\forall x\, A)^\neg := \neg\neg\forall x\, A^\neg$,
- $(\exists x\, A)^\neg := \neg\neg\exists x\, A^\neg$.

Lemma 5.15. *For every formula* A*, we have* $\mathsf{PA} \vdash A$ *iff* $\mathsf{HA} \vdash A^\neg$.

Proof. The direction from right to left is trivial. For the other way around, one may use induction on the proof $\mathsf{PA} \vdash A$. For details see [TvD88]. ❑

Lemma 5.16. *For every* Σ_1*-formula* A*, we have* $\mathsf{HA} \vdash A^{\neg} \leftrightarrow \neg\neg A$.

Proof. Easy by use of the decidability of Δ_0-formulas in HA (Lemma 3.12). ❑

Lemma 5.17. *For every* $A \in \mathcal{L}_\Box$*, we have* $\mathsf{iH}_\sigma\underline{\mathsf{P}} \vdash A$ *iff* $\mathsf{iH}_\sigma \vdash A^{\neg\uparrow}$.

Proof. The direction from right to left holds by the classically valid $A \leftrightarrow A^{\neg\uparrow}$. For the other way around, one must use induction on the length of the proof $\mathsf{iH}_\sigma\underline{\mathsf{P}} \vdash A$. All cases are easy and left to the reader. ❑

Lemma 5.18. *For every* $A \in \mathcal{L}_\Box$*, recursively axiomatizable theory* T *and any* Σ_1*-substitution* σ*, we have* $\mathsf{HA} \vdash (\sigma_{\mathsf{T}}(A))^{\neg} \leftrightarrow \sigma_{\mathsf{T}}(A^{\neg\uparrow})$.

Proof. We use induction on the complexity of A. All cases are simple. For atomic and boxed cases, use Lemma 5.16. ❑

Theorem 5.19. $\mathsf{iH}_\sigma\underline{\mathsf{P}} = \mathcal{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{PA}) \leq \mathcal{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{HA}) = \mathsf{iH}_\sigma$.

Proof. By Theorem 5.12 and Theorem 5.7 we have $\mathsf{iH}_\sigma\underline{\mathsf{P}} = \mathcal{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{PA})$ and $\mathcal{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{HA}) = \mathsf{iH}_\sigma$. We must show $\mathcal{AC}_{\Sigma_1}(\mathsf{iH}_\sigma\underline{\mathsf{P}}; \mathsf{HA}, \mathsf{PA}) \leq_{f,\bar{f}} \mathcal{AC}_{\Sigma_1}(\mathsf{iH}_\sigma; \mathsf{HA}, \mathsf{HA})$. Given $A \in \mathcal{L}_\Box$, define $f(A) := A^{\neg\uparrow}$ and observe by Lemma 5.17 we have R1 (see Definition 4.1). Also define $\bar{f}_A$ as identity function. Then by Lemmas 5.15 and 5.18 the condition R2 holds. ❑

Theorem 5.20. $\mathsf{iH}_\sigma = \mathcal{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{HA}) \leq \mathcal{PL}_{\Sigma_1}(\mathsf{HA}, \mathbb{N}) = \mathsf{iH}_\sigma\underline{\mathsf{SP}}$.

Proof. By Theorems 5.7 and 5.13 we have $\mathsf{iH}_\sigma = \mathcal{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{HA})$ and $\mathcal{PL}_{\Sigma_1}(\mathsf{HA}, \mathbb{N}) = \mathsf{iH}_\sigma\underline{\mathsf{SP}}$. We must show $\mathcal{AC}_{\Sigma_1}(\mathsf{iH}_\sigma; \mathsf{HA}, \mathsf{HA}) \leq_{f,\bar{f}} \mathcal{AC}_{\Sigma_1}(\mathsf{iH}_\sigma\underline{\mathsf{SP}}; \mathsf{HA}, \mathbb{N})$. Given $A \in \mathcal{L}_\Box$, define $f(A) = \Box A$ and $\bar{f}_A$ as identity function.

R1. Let $\mathsf{iH}_\sigma\underline{\mathsf{SP}} \vdash \Box A$. By soundness of $\mathsf{iH}_\sigma\underline{\mathsf{SP}} = \mathcal{PL}_{\Sigma_1}(\mathsf{HA}, \mathbb{N})$, for every Σ_1-substitution σ we have $\mathbb{N} \models \sigma_{\mathsf{HA}}(\Box A)$ and hence $\mathsf{HA} \vdash \sigma_{\mathsf{HA}}(A)$. Then by arithmetical completeness of $\mathcal{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{HA})$, we have $\mathsf{iH}_\sigma \vdash A$.
One also may prove this item with a direct propositional argument. For simplicity reasons, we chose the indirect way.

R2. Let $\mathbb{N} \not\models \sigma_{\mathsf{HA}}(\Box A)$. Then $\mathsf{HA} \nvdash \sigma_{\mathsf{HA}}(A)$, as desired. ❑

6 Relative Σ_1-provability logics for HA^*

The σ_1-provability logic of HA^*, $\mathcal{PL}_{\Sigma_1}(\mathsf{HA}^*, \mathsf{HA}^*)$, is already characterized [AM19]. In this section, we characterize the Σ_1-provability logic of HA^*, relative in PA and $\mathbb{N}$. We also show that the following reductions hold:

Each arrow in the above diagram, indicates a reduction of the completeness of the left hand side to the right one. Note that the diagram of the first row is already known by Theorems 5.19 and 5.20.

Theorem 6.1. $\mathsf{iH}_\sigma\underline{\mathsf{SP}}^* = \mathcal{PL}_{\Sigma_1}(\mathsf{HA}^*, \mathbb{N}) \leq \mathcal{PL}_{\Sigma_1}(\mathsf{HA}, \mathbb{N}) = \mathsf{iH}_\sigma\underline{\mathsf{SP}}$. (*See Definition 5.11*)

Proof. By Theorem 5.13 we have $\mathcal{PL}_{\Sigma_1}(\mathsf{HA}, \mathbb{N}) = \mathsf{iH}_\sigma\underline{\mathsf{SP}}$. It is enough to prove the arithmetical soundness $\mathcal{AS}_{\Sigma_1}(\mathsf{iH}_\sigma\underline{\mathsf{SP}}^*; \mathsf{HA}^*, \mathbb{N})$ and the reduction $\mathcal{AC}_{\Sigma_1}(\mathsf{iH}_\sigma\underline{\mathsf{SP}}^*; \mathsf{HA}^*, \mathbb{N}) \leq \mathcal{AC}_{\Sigma_1}(\mathsf{iH}_\sigma\underline{\mathsf{SP}}; \mathsf{HA}, \mathbb{N})$.
$\mathcal{AS}_{\Sigma_1}(\mathsf{iH}_\sigma\underline{\mathsf{SP}}^*; \mathsf{HA}^*, \mathbb{N})$: Let $\mathsf{iH}_\sigma\underline{\mathsf{SP}}^* \vdash A$ and σ is a Σ_1-substitution. Then $\mathsf{iH}_\sigma\underline{\mathsf{SP}} \vdash A^{\Box\downarrow}$, and then by arithmetical soundness of $\mathsf{iH}_\sigma\underline{\mathsf{SP}}$ Theorem 5.13, we have $\mathbb{N} \models \sigma_{\mathsf{HA}}(A^{\Box\downarrow})$. Hence Lemma 3.19 implies $\mathbb{N} \models \sigma_{\mathsf{HA}^*}(A)$, as desired.
For the proof of $\mathcal{AC}_{\Sigma_1}(\mathsf{iH}_\sigma\underline{\mathsf{SP}}^*; \mathsf{HA}^*, \mathbb{N}) \leq_{f,\bar{f}} \mathcal{AC}_{\Sigma_1}(\mathsf{iH}_\sigma\underline{\mathsf{SP}}; \mathsf{HA}, \mathbb{N})$, define $f(A) := A^{\Box\downarrow}$ and $\bar{f}_A$ as identity function.

R1. Let $\mathsf{iH}_\sigma\underline{\mathsf{SP}} \vdash A^{\Box\downarrow}$. Then by definition we have $\mathsf{iH}_\sigma\underline{\mathsf{SP}}^* \vdash A$.

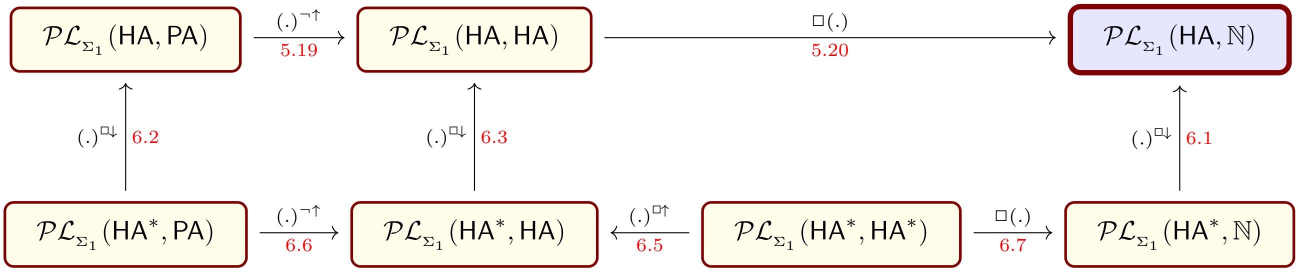

Diagram 2: Reductions for relative provability logics of HA^*

R2. Let $\mathbb{N} \not\models \sigma_{\mathsf{HA}^*}(A^{\Box\downarrow})$. Hence by Lemma 3.19 $\mathbb{N} \not\models \sigma_{\mathsf{HA}}(A)$, as desired. ❑

Theorem 6.2. $\mathsf{iH}_\sigma\underline{\mathsf{P}}^* = \mathcal{PL}_{\Sigma_1}(\mathsf{HA}^*,\mathsf{PA}) \le \mathcal{PL}_{\Sigma_1}(\mathsf{HA},\mathsf{PA}) = \mathsf{iH}_\sigma\underline{\mathsf{P}}$. (*See Definition 5.11*)

Proof. Similar to the proof of Theorem 6.1 and left to the reader. ❑

Theorem 6.3. $\mathsf{iH}^*_\sigma = \mathcal{PL}_{\Sigma_1}(\mathsf{HA}^*,\mathsf{HA}) \le \mathcal{PL}_{\Sigma_1}(\mathsf{HA},\mathsf{HA}) = \mathsf{iH}_\sigma$. (*See Definition 3.28*)

Proof. Similar to the proof of Theorem 6.1 and left to the reader. ❑

Lemma 6.4. *For every $A \in \mathcal{L}_\Box$ we have $\mathsf{iH}^{**}_\sigma \vdash \mathsf{A}$ iff $\mathsf{iH}^*_\sigma \vdash A^{\Box\uparrow}$.* (*See Definition 3.28*)

Proof. We have the following equivalents: $\mathsf{iH}^{**}_\sigma \vdash \mathsf{A}$ iff $\mathsf{iH}_\sigma \vdash A^\Box$ iff (by Remark 3.3) $\mathsf{iH}_\sigma \vdash (A^{\Box\uparrow})^{\Box\downarrow}$ iff $\mathsf{iH}^*_\sigma \vdash A^{\Box\uparrow}$. ❑

Theorem 6.5. $\mathsf{iH}^{**}_\sigma = \mathcal{PL}_{\Sigma_1}(\mathsf{HA}^*,\mathsf{HA}^*) \le \mathcal{PL}_{\Sigma_1}(\mathsf{HA}^*,\mathsf{HA}) = \mathsf{iH}^*_\sigma$.

Proof. By Theorem 6.3 we have $\mathcal{PL}_{\Sigma_1}(\mathsf{HA}^*,\mathsf{HA}) = \mathsf{iH}^*_\sigma$. It is enough to prove the arithmetical soundness $\mathcal{AS}_{\Sigma_1}(\mathsf{iH}^{**}_\sigma;\mathsf{HA}^*,\mathsf{HA}^*)$ and the reduction $\mathcal{AC}_{\Sigma_1}(\mathsf{iH}^{**}_\sigma;\mathsf{HA}^*,\mathsf{HA}^*) \le \mathcal{AC}_{\Sigma_1}(\mathsf{iH}^*_\sigma;\mathsf{HA}^*,\mathsf{HA})$.
$\mathcal{AS}_{\Sigma_1}(\mathsf{iH}^{**}_\sigma;\mathsf{HA}^*,\mathsf{HA}^*)$: Let $\mathsf{iH}^{**}_\sigma \vdash \mathsf{A}$ and σ is a Σ_1-substitution. Then $\mathsf{iH}_\sigma \vdash A^\Box$, and then by arithmetical soundness of iH_σ Theorem 5.7, we have $\mathsf{HA} \vdash \sigma_{\mathsf{HA}}(A^\Box)$. Hence Lemma 3.18 implies $\mathsf{HA} \vdash \sigma_{\mathsf{HA}^*}(A)^{\mathsf{HA}}$, which implies $\mathsf{HA}^* \vdash \sigma_{\mathsf{HA}^*}(A)$.
For the proof of $\mathcal{AC}_{\Sigma_1}(\mathsf{iH}_\sigma\underline{\mathsf{SP}}^*;\mathsf{HA}^*,\mathbb{N}) \le_{f,\bar{f}} \mathcal{AC}_{\Sigma_1}(\mathsf{iH}_\sigma\underline{\mathsf{SP}};\mathsf{HA},\mathbb{N})$, define $f(A) := A^{\Box\uparrow}$ and $\bar{f}_A$ as identity function.

R1. Let $\mathsf{iH}^*_\sigma \vdash A^{\Box\uparrow}$. Then by Lemma 6.4 we have $\mathsf{iH}^{**}_\sigma \vdash \mathsf{A}$, as desired.

R2. Let $\mathsf{HA} \nvdash \sigma_{\mathsf{HA}^*}(A^{\Box\uparrow})$. Hence by Lemma 3.17 we have $\mathsf{HA} \nvdash (\sigma_{\mathsf{HA}^*}(A))^{\mathsf{HA}}$, which implies $\mathsf{HA}^* \nvdash \sigma_{\mathsf{HA}^*}(A)$, as desired. ❑

Theorem 6.6. $\mathsf{iH}_\sigma\underline{\mathsf{P}}^* = \mathcal{PL}_{\Sigma_1}(\mathsf{HA}^*,\mathsf{PA}) \le \mathcal{PL}_{\Sigma_1}(\mathsf{HA}^*,\mathsf{HA}) = \mathsf{iH}^*_\sigma$.

Proof. $\mathsf{iH}_\sigma\underline{\mathsf{P}}^* = \mathcal{PL}_{\Sigma_1}(\mathsf{HA}^*,\mathsf{PA})$ and $\mathsf{iH}^*_\sigma = \mathcal{PL}_{\Sigma_1}(\mathsf{HA}^*,\mathsf{HA})$, by Theorems 6.2 and 6.3 holds. Given A, define $f(A) := A^{\neg\uparrow}$ and $\bar{f}_A$ as identity function.

R1. By definition of $\mathsf{iH}_\sigma\underline{\mathsf{P}}^*$, we have $\mathsf{iH}_\sigma\underline{\mathsf{P}}^* \vdash A$ iff $\mathsf{iH}_\sigma\underline{\mathsf{P}} \vdash A^{\Box\downarrow}$. The latter, by Lemma 5.17 is equivalent to $\mathsf{iH}_\sigma \vdash (A^{\Box\downarrow})^{\neg\uparrow}$. Since $(A^{\Box\downarrow})^{\neg\uparrow} = (A^{\neg\uparrow})^{\Box\downarrow}$, the latter is equivalent to $\mathsf{iH}^*_\sigma \vdash A^{\neg\uparrow}$.

R2. By Lemmas 5.15 and 5.18. ❑

Theorem 6.7. $\mathsf{iH}^{**}_\sigma = \mathcal{PL}_{\Sigma_1}(\mathsf{HA}^*,\mathsf{HA}^*) \le \mathcal{PL}_{\Sigma_1}(\mathsf{HA}^*,\mathbb{N}) = \mathsf{iH}_\sigma\underline{\mathsf{SP}}^*$.

Proof. By Theorems 6.1 and 6.5 we have $\mathcal{PL}_{\Sigma_1}(\mathsf{HA}^*, \mathbb{N}) = \mathsf{iH}_\sigma\underline{\mathsf{SP}}^*$ and $\mathsf{iH}_\sigma^{**} = \mathcal{PL}_{\Sigma_1}(\mathsf{HA}^*, \mathsf{HA}^*)$. We must show $\mathcal{AC}_{\Sigma_1}(\mathsf{iH}_\sigma^{**}; \mathsf{HA}^*, \mathsf{HA}^*) \leq_{\mathsf{f},\bar{\mathsf{f}}} \mathcal{AC}_{\Sigma_1}(\mathsf{iH}_\sigma\underline{\mathsf{SP}}^*; \mathsf{HA}, \mathbb{N})$. Given $A \in \mathcal{L}_\Box$, define $f(A) = \Box A$ and $\bar{f}_A$ as identity function.

R1. Let $\mathsf{iH}_\sigma\underline{\mathsf{SP}}^* \vdash \Box A$. By soundness of $\mathsf{iH}_\sigma\underline{\mathsf{SP}}^* = \mathcal{PL}_{\Sigma_1}(\mathsf{HA}^*, \mathbb{N})$, for every Σ_1-substitution σ we have $\mathbb{N} \models \sigma_{\mathsf{HA}^*}(\Box A)$ and hence $\mathsf{HA}^* \vdash \sigma_{\mathsf{HA}^*}(A)$. Then by arithmetical completeness of $\mathcal{PL}_{\Sigma_1}(\mathsf{HA}^*, \mathsf{HA}^*)$, we have $\mathsf{iH}_\sigma^{**} \vdash \mathsf{A}$.
One also may prove this item with a direct propositional argument. For simplicity reasons, we chose the indirect way.

R2. If $\mathbb{N} \not\models \sigma_{\mathsf{HA}^*}(\Box A)$ evidently we have $\mathsf{HA}^* \nvdash \sigma_{\mathsf{HA}^*}(A)$. ❑

7 Relative provability logics for PA

In this section, we characterize $\mathcal{PL}(\mathsf{PA}, \mathsf{HA})$ and $\mathcal{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{HA})$, the provability logic and Σ_1-provability logic of PA relative in HA. We show that $\mathcal{PL}(\mathsf{PA}, \mathsf{HA}) = \mathsf{iGL}\overline{\mathsf{P}}$ and $\mathcal{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{HA}) = \mathsf{iGL}\overline{\mathsf{P}}\mathsf{C_a}$. Also we show that all of the six (Σ_1-) provability logics of PA relative in $\mathsf{PA}, \mathsf{HA}, \mathbb{N}$ are reducible to $\mathcal{PL}_{\Sigma_1}(\mathsf{HA}, \mathbb{N})$:

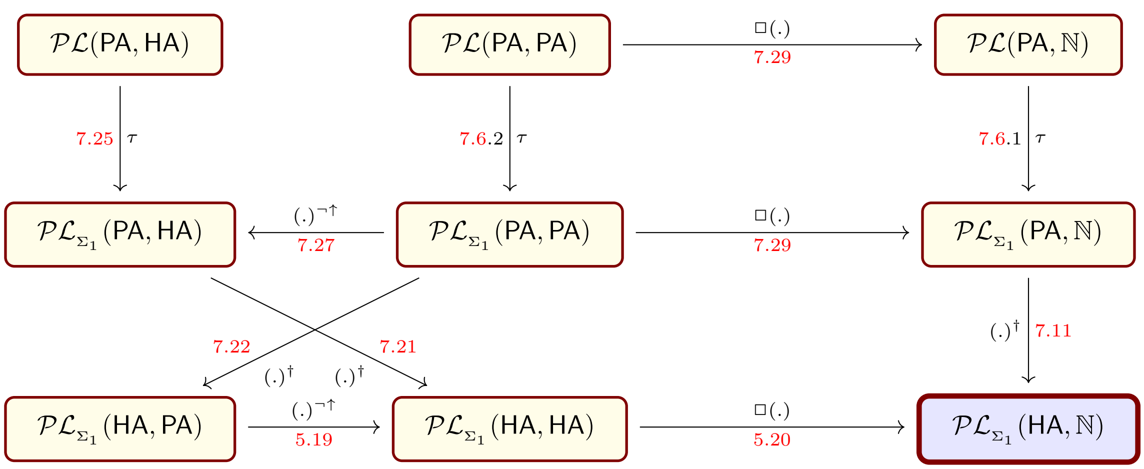

Diagram 3: Reductions for relative provability logics of PA

Let us first review some well-known results:

Theorem 7.1. *We have the following provability logics:*

- GL *is the provability logic of* PA, *i.e.* $\mathcal{PL}(\mathsf{PA}, \mathsf{PA}) = \mathsf{GL}$. *[Sol76]*
- $\mathsf{GL\underline{S}}$ *is the truth provability logic of* PA, *i.e.* $\mathcal{PL}(\mathsf{PA}, \mathbb{N}) = \mathsf{GL\underline{S}}$. *[Sol76]*
- $\mathsf{GLC_a}$ *is the* Σ_1*-provability logic of* PA, *i.e.* $\mathcal{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{PA}) = \mathsf{GLC_a}$. *[Vis82]*
- $\mathsf{GL\underline{S}C_a}$ *is the truth* Σ_1*-provability logic of* PA, *i.e.* $\mathcal{PL}_{\Sigma_1}(\mathsf{PA}, \mathbb{N}) = \mathsf{GL\underline{S}C_a}$. *[Vis82]*

Definition 7.2. A propositional modal substitution τ is called $(.)^{\Box\downarrow}$-substitution, if for every atomic variable p, there is some B such that $\mathsf{iK4} + \mathsf{CP_a} \vdash \tau(p) \leftrightarrow B^{\Box\downarrow}$ and $\mathsf{iK4} \vdash \boxdot B^{\Box\downarrow} \leftrightarrow B^{\Box}$.

Lemma 7.3. *For every* $(.)^{\Box\downarrow}$*-substitution* τ *and every modal proposition* A*, we have* $\mathsf{iK4V} \vdash \tau(A^{\Box}) \leftrightarrow \tau(A)^{\Box}$ *and* $\mathsf{iK4V} \vdash \tau(A^{\Box\downarrow}) \leftrightarrow \tau(A)^{\Box\downarrow}$.

Proof. First by induction on the complexity of B we show $\mathsf{iK4V} \vdash \tau(B^{\Box}) \leftrightarrow \tau(B)^{\Box}$. All cases are easy, except for atomic B, which holds by existence of some C such that $\mathsf{iK4V} \vdash \tau(B) \leftrightarrow C^{\Box\downarrow}$ and $\mathsf{iK4} \vdash \boxdot C^{\Box\downarrow} \leftrightarrow C^{\Box}$.
Then we use induction on the complexity of A to deduce the second assertion of this lemma. The only non-trivial cases are atomic and boxed cases:

- A is atomic. Since $\mathsf{iK4} \vdash B^{\Box\downarrow} \leftrightarrow (B^{\Box\downarrow})^{\Box\downarrow}$ for every B, and $\mathsf{iK4V} \vdash \tau(A) \leftrightarrow B^{\Box\downarrow}$, we have the desired result.
- $A = \Box B$. Easily deduced by $\mathsf{iK4V} \vdash \tau(B^{\Box}) \leftrightarrow \tau(B)^{\Box}$.

❑

The following remark, will be helpful for later reductions of provability logics in section 8.

Remark 7.4. For every modal proposition A, $\mathsf{GL} \vdash A$ ($\mathsf{GL\underline{S}} \vdash A$) iff for every $(.)^{\Box\downarrow}$-substitution τ we have $\mathsf{GLC_a} \vdash \tau(A)$ ($\mathsf{GL\underline{S}C_a} \vdash \tau(A)$).

Proof. See [AM15, Lemmas. 3.1 and 3.3] . ❑

Lemma 7.5. *For every* $A \in \mathcal{L}_\Box$*,*

- $\mathsf{GL\underline{S}} \vdash \Box A$ *iff* $\mathsf{GL} \vdash A$*,*
- $\mathsf{GL\underline{S}C_a} \vdash \Box A$ *iff* $\mathsf{GLC_a} \vdash A$*.*

Proof. The proof of second item is similar to the first one. Here we only treat the first item. Obviously, $\mathsf{GL} \vdash A$ implies $\mathsf{GL\underline{S}} \vdash \Box A$. For a direct proof of the other way around, one may use of Smorýnski's operation. However, now that we enjoy the arithmetical soundness of $\mathcal{PL}(\mathsf{PA}, \mathbb{N}) = \mathsf{GL\underline{S}}$, from $\mathsf{GL\underline{S}} \vdash \Box A$ for every σ we have $\mathbb{N} \models \sigma_{\mathsf{PA}}(\Box A)$ and hence $\mathsf{PA} \vdash \sigma_{\mathsf{PA}}(A)$. From the arithmetical completeness of $\mathsf{GL} = \mathcal{PL}(\mathsf{PA}, \mathsf{PA})$, we get $\mathsf{GL} \vdash A$. ❑

In the following theorem, we will show that $\mathsf{GL\underline{S}C_a}$ is the hardest provability logic among GL, $\mathsf{GLC_a}$, $\mathsf{GL\underline{S}}$ and $\mathsf{GL\underline{S}C_a}$.

Theorem 7.6. *We have the following reductions:*

1. $\mathcal{PL}(\mathsf{PA}, \mathbb{N}) \leq \mathcal{PL}_{\Sigma_1}(\mathsf{PA}, \mathbb{N})$,
2. $\mathcal{PL}(\mathsf{PA}, \mathsf{PA}) \leq \mathcal{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{PA})$.

Proof. We prove each item separately:

1. We must show that $\mathcal{AC}(\mathsf{GL\underline{S}}; \mathsf{PA}, \mathbb{N}) \leq_{f,\bar{f}} \mathcal{AC}_{\Sigma_1}(\mathsf{GL\underline{S}C_a}; \mathsf{PA}, \mathbb{N})$. Consider some $A \in \mathcal{L}_\Box$. If $\mathsf{GL\underline{S}} \nvdash A$, by Remark 7.4, there exists some $\mathcal{L}_\Box$-substitution τ such that $\mathsf{GL\underline{S}C_a} \nvdash \tau(A)$. Let
$$f(A) := \begin{cases} \tau(A) & : \mathsf{GL\underline{S}} \nvdash A \\ A & : \text{otherwise} \end{cases}$$
Hence R1 (Definition 4.1) holds. Also let $\bar{f}_A(\sigma) := \sigma_{\mathsf{PA}} \circ \tau$, which belongs to $[\![\sigma]\!]$. Then obviously R2 holds.
2. Similar to first item and left to the reader. ❑

7.1 Reducing $\mathcal{PL}_{\Sigma_1}(\mathsf{PA},\mathbb{N})$ to $\mathcal{PL}_{\Sigma_1}(\mathsf{HA},\mathbb{N})$

In this subsection, we illustrate how to reduce the arithmetical completeness of $\mathsf{GL\underline{S}C_a}$ to that of $\mathsf{iH}_\sigma\underline{\mathsf{SP}}$. First some definitions and lemmas:

Definition 7.7. For a modal proposition A let $A^\ddagger$ indicates the classically equivalent formula of the form

$$A^\ddagger := \bigwedge_i B_i \to C_i \quad \text{in which} \quad B_i = \bigwedge_j E^\ddagger_{i,j} \text{ and } C_i = \bigvee_j F^\ddagger_{i,j}$$

and $E_{i,j}, F_{i,j}$ are atomic or boxed formulas. Also for atomic p we have $p\ddagger = p$ and $(\Box E)^\ddagger = \Box(E^\ddagger)$. Then define $A^\dagger$ in this way:

- $(.)^\dagger$ commutes with $\vee, \wedge, \to$,
- $p^\dagger = p$ for atomic p,
- $(\Box A)^\dagger = \Box A^\ddagger$

Lemma 7.8. *For every modal proposition A and arithmetical substitution α, we have*

$$\mathsf{HA} \vdash \alpha_{\mathsf{HA}}(A^\dagger) \leftrightarrow \alpha_{\mathsf{PA}}(A)$$

Proof. Easy and left to the reader. ❑

Lemma 7.9. *For every $A \in \mathcal{L}_\Box$, if $\mathsf{iH}_\sigma\underline{\mathsf{SP}} \vdash A^\dagger$ then $\mathsf{GL\underline{S}C_a} \vdash A$.*

Proof. Let $\mathsf{GL\underline{S}C_a} \nvdash A$. Since in classical logic we have $A \leftrightarrow A^\dagger$, then $\mathsf{GL\underline{S}C_a} \nvdash A^\dagger$. Hence by $\mathcal{AC}_{\Sigma_1}(\mathsf{GL\underline{S}C_a}; \mathsf{PA}, \mathbb{N})$ from 7.1, we have some Σ_1-substitution σ such that $\mathbb{N} \not\models \sigma_{\mathsf{PA}}(A^\dagger)$. Then Lemma 7.8 implies $\mathbb{N} \not\models \sigma_{\mathsf{HA}}(A^\dagger)$, and hence by arithmetical soundness of $\mathsf{iH}_\sigma\underline{\mathsf{SP}}$ (Theorem 5.13) we have $\mathsf{iH}_\sigma\underline{\mathsf{SP}} \nvdash A^\dagger$, as desired. ❑

Lemma 7.10. *For every $A \in \mathcal{L}_\Box$, if $\mathsf{iH}_\sigma\underline{\mathsf{P}} \vdash A^\dagger$ then $\mathsf{GLC_a} \vdash A$.*

Proof. Let $\mathsf{GLC_a} \nvdash A$. Since in classical logic we have $A \leftrightarrow A^\dagger$, then $\mathsf{GLC_a} \nvdash A^\dagger$. Hence by $\mathcal{AC}_{\Sigma_1}(\mathsf{GLC_a}; \mathsf{PA}, \mathsf{PA})$ from 7.1, we have some Σ_1-substitution σ such that $\mathsf{PA} \nvdash \sigma_{\mathsf{PA}}(A^\dagger)$. Then Lemma 7.8 implies $\mathsf{PA} \nvdash \sigma_{\mathsf{HA}}(A^\dagger)$, and hence by arithmetical soundness of $\mathsf{iH}_\sigma\underline{\mathsf{P}}$ (Theorem 5.12) we have $\mathsf{iH}_\sigma\underline{\mathsf{P}} \nvdash A^\dagger$, as desired. ❑

Theorem 7.11. $\mathsf{GL\underline{S}C_a} = \mathcal{PL}_{\Sigma_1}(\mathsf{PA},\mathbb{N}) \leq \mathcal{PL}_{\Sigma_1}(\mathsf{HA},\mathbb{N}) = \mathsf{iH}_\sigma\underline{\mathsf{SP}}$.

Proof. By Theorems 5.13 and 7.1 we have $\mathsf{iH}_\sigma\underline{\mathsf{SP}} = \mathcal{PL}_{\Sigma_1}(\mathsf{HA},\mathbb{N})$ and $\mathsf{GL\underline{S}C_a} = \mathcal{PL}_{\Sigma_1}(\mathsf{PA},\mathbb{N})$. For the reduction, let $f(A) := A^\dagger$ and $\bar{f}_A$ as identity function.

R1. If $\mathsf{iH}_\sigma\underline{\mathsf{SP}} \vdash A^\dagger$, by Lemma 7.9 we have $\mathsf{GL\underline{S}C_a} \vdash A$.

R2. Holds by Lemma 7.8. ❑

7.2 Kripke Semantics

Let $\mathsf{Suc}_\mathcal{K}$ or simply Suc, when no confusion is likely, indicates the set of all $\sqsubset$-accessible nodes in the Kripke model $\mathcal{K}$.

Theorem 7.12. $\mathsf{iGL\overline{P}}$ *is sound and complete for semi-perfect* Suc*-classical* $\sqsubset$*-branching Kripke models.*

Proof. The soundness is easy and left to the reader. For the completeness, we first show the completeness for finite brilliant irreflexive transitive Suc-classical Kripke models. Let $\mathsf{iGL\overline{P}} \nvdash A$. Let

$$X := \{B, \neg B, B \vee \neg B : B \in \mathsf{Sub}(A)\} \cup \{\bot\}$$

and define the Kripke model $\mathcal{K} = (K, \preccurlyeq, \sqsubset, V)$ as follows:

- K is the family of all X-saturated sets with respect to $\mathsf{iGL\overline{P}}$.
- $\alpha \preccurlyeq \beta$ iff $\alpha \subseteq \beta$.
- $\alpha \sqsubset \beta$ iff β is a maximally consistent set and $\{B, \Box B : \Box B \in \alpha\} \subseteq \beta$ and there is some $\Box B \in \beta \setminus \alpha$.

It is straightforward to show that $\mathcal{K}$ is actually a finite brilliant irreflexive Suc-classical Kripke model, and we leave all of them to the reader.

It is enough to show that $\mathcal{K}, \alpha \Vdash B$ iff $B \in \alpha$ for every $\alpha \in K$ and $B \in X$. Then we may use Lemma 3.37 and find some α such that $\mathcal{K}, \alpha \nVdash A$. We use induction on the complexity of $B \in X$. All inductive steps are trivial, except for $B = \Box C$. If $\Box C \in \alpha$ and $\alpha \sqsubset \beta$, then by definition, $C \in \beta$ and hence by induction hypothesis $\beta \Vdash C$. This implies $\alpha \Vdash \Box C$. For the other way around, let $\Box C \notin \alpha$. Consider the set $\Delta := \{E, \Box E : \Box E \in \alpha\}$. If $\mathsf{GL} \vdash \bigwedge \Delta \to (\Box C \to C)$, then $\mathsf{iGL} + \Box\mathsf{PEM} \vdash \Box(\bigwedge \Delta) \to \Box C$. Since $\mathsf{iGL\overline{P}} \vdash \alpha \to \Box \bigwedge \Delta$ and $\mathsf{iGL\overline{P}} \vdash \Box\mathsf{PEM}$, we have $\mathsf{iGL\overline{P}} + \alpha \vdash \Box C$ and hence $\Box C \in \alpha$, a contradiction. Hence we have $\mathsf{GL} \nvdash (\bigwedge \Delta \wedge \Box C) \to C$. Then by Lemma 3.37 there is some X-saturated set $\beta \supseteq \Delta \cup \{\Box C\} \cup \{E \vee \neg E : E \in \mathsf{Sub}(A)\}$ such that $C \notin \beta$. Hence $\beta \sqsupset \alpha$ and $\beta \nVdash C$. Then $\alpha \nVdash \Box C$, as desired.

Next we use the construction method [Iem01], to fulfil the other conditions: $\sqsubset$-branching, neat and tree. Let $\mathcal{K}_t := (K_t, \preccurlyeq_t, \sqsubset_t, V_t)$ as follows:

- K_t is the set of all finite sequences of pairs $r := \langle (\alpha_0, a_0), \ldots (\alpha_n, a_n) \rangle$ such that for any $i \leq n$: (1) $\alpha_i \in K$, (2) $a_i \in \{0, 1\}$, (3) for $i < n$ either we have $\alpha_i \prec \alpha_{i+1}$ or $\alpha_i \sqsubset \alpha_{i+1}$. Let $f_1(r)$ and $f_2(r)$ indicate the left and right elements in the final element of the sequence r. In other words, we let $(f_1(r), f_2(r))$ be the final element of the sequence r.
- $r \preccurlyeq_t s$ iff r is an initial segment of s and $f_1(r) \preccurlyeq f_1(s)$.
- $r \sqsubset_t s$ iff r is an initial segment of $s = \langle (\alpha_0, a_0), \ldots, (\alpha_n, a_n) \rangle$, e.g. $r = \langle (\alpha_0, a_0), \ldots, (\alpha_k, a_k) \rangle$ for some $k < n$ and $\alpha_i \sqsubset \alpha_{i+1}$ for some $k \leq i < n$.
- $r \, V_t \, p$ iff $f_1(r) \, V \, p$.

It is straightforward to show that $\mathcal{K}_t$ is semi-perfect $\sqsubset$-branching Suc-classical Kripke model and for every $r \in K_t$ and formula B we have

$$\mathcal{K}_t, r \Vdash B \quad \Longleftrightarrow \quad \mathcal{K}, f(r) \Vdash B.$$

❑

Theorem 7.13. $\mathsf{iGL\overline{P}C_a}$ *is sound and complete for semi-perfect* Suc*-classical atom-complete Kripke models.*

Proof. The proof is almost identical to the one for Theorem 7.12. We only explain the differences here. Define

$$X := \{B, \neg B, B \vee \neg B : B \in \mathsf{Sub}(A)\} \cup \{\bot\} \cup \{\Box p : p \in \mathsf{Sub}(A) \text{ and } p \text{ is atomic}\}$$

and K, the set of the nodes of Kripke model, is defined as the set of all X-saturated sets with respect to $\mathsf{iGL\overline{P}C_a}$. We show that every $\alpha \in K$ is atom-complete. Let p be an atomic variable such that $\alpha \Vdash p$. Hence $p \in \alpha$ which implies $p \in \mathsf{Sub}(A)$, and since $\mathsf{iGL\overline{P}C_a} \vdash p \to \Box p$ and α is closed under deduction, we have $\Box p \in \alpha$. Then $\alpha \Vdash \Box p$ and hence for every $\beta \sqsupset \alpha$ we have $\beta \Vdash p$, as desired. ❑

7.3 Arithmetical Completeness

Theorem 7.14. $\mathsf{iGL\overline{P}C_a}$ *is the relative* Σ_1*-provability logic of* PA *in* HA*, i.e.* $\mathcal{PL}_{\Sigma_1}(\mathsf{PA},\mathsf{HA}) = \mathsf{iGL\overline{P}C_a}$.

Proof. The soundness is straightforward and left to the reader. For the completeness part, let $\mathsf{iGL\overline{P}C_a} \nvdash A$. Then by Theorem 7.13, there is some semi-perfect atom-complete Suc-classical Kripke model $\mathcal{K} = (K, \preccurlyeq, \sqsubset, V)$ such that $\mathcal{K}, \alpha_0 \nVdash A$ for some $\alpha_0 \in K$. Without loss of generality, we may assume that $K = (\alpha_0 \preccurlyeq) \cup (\alpha_0 \sqsubset)$. Let $\mathcal{K}' = (K', \preccurlyeq', \sqsubset', V')$ indicates the Smorýnski's extension of $\mathcal{K}$ at α_0 with the fresh node α_1. For the simplicity of notations, we may use $\preccurlyeq$ and $\sqsubset$ instead of $\preccurlyeq'$ and $\sqsubset'$. Define the recursive function F as follows. Since K' is a finite set, we might assign a unique number $\bar{\alpha}$ to each node α and speak about K' and its relationships $\preccurlyeq$ and $\sqsubset$ inside the language of arithmetic. For simplicity of notations, we may simply use $\alpha \preccurlyeq \beta$ and $\alpha \sqsubset \beta$ corresponding to its equivalent arithmetical formula.

Define $F(0) := \alpha_1$ and

$$F(n+1) := \begin{cases} \beta & : F(n) \sqsubset \beta \text{ and } r(\beta, n+1) < n+1 \text{ and } (n)_0 = \beta \\ \beta & : F(n) \prec \beta \text{ and } F(n) \not\sqsubset \beta \text{ and } \ F(r(\beta, n+1)) = \alpha_1 \\ & \quad \text{and } r(\beta, n+1) < r(F(n), n+1) \text{ and } (n)_0 = \beta \\ F(n) & : \text{otherwise} \end{cases}$$

in which $L = \beta$ is shorthand for $\exists x \forall y \geq x (F(y) = F(x))$, $(n)_0$ is the exponent of 2 in n and

$$r(\alpha, n) := \mathsf{min}\left(\{x \in \mathbb{N} : \exists t \leq n \, \mathsf{Proof}_{\mathsf{PA}_x}(t, \ulcorner L \neq \alpha \urcorner)\} \cup \{n\}\right)$$

Note that $r(\alpha, n) < n$ implies $\Box^+(L \neq \alpha)$. F is a provably total recursive function in HA, i.e. $F(x) = y$ could be expressed as a Σ_1-formula in the language of arithmetic and all of its expected properties are provable in HA. Hence we may use the function symbol F in the language of arithmetic.

Define the arithmetical substitution $\sigma(p)$ in this way:

$$\sigma(p) := \bigvee_{\mathcal{K}, \alpha \Vdash p} \exists x \, F(x) = \alpha$$

Consider the triple $\mathcal{I} := (K^*, \preccurlyeq^*, T)$ as follows:

- $K^* := \{\alpha \in K : \nexists \beta \in K(\beta \sqsubset \alpha)\}$.
- $\alpha \preccurlyeq^* \beta$ iff $\alpha \preccurlyeq \beta$ for every $\alpha, \beta \in K^*$. Again, by abuse of notations, we use $\preccurlyeq$ instead of $\preccurlyeq^*$.
- $T(\alpha) := \mathsf{PA} + (L = \alpha)$.

By Theorem 3.24 and Lemma 7.16, we have some first-order Kripke model $\mathcal{K}^* = (K^*, \preccurlyeq, \mathfrak{M})$ such that $\mathcal{K}^* \Vdash \mathsf{HA}$ and $\mathcal{K}^*, \alpha \models T(\alpha)$. By Lemma 3.21

$$\mathcal{K}^*, \alpha \Vdash \exists x \, F(x) = \beta \quad \Longrightarrow \quad \beta \preccurlyeq \alpha \tag{7.1}$$

Hence by Lemma 3.21, for every $\alpha \in K^*$

$$\mathcal{K}^*, \alpha \Vdash \sigma_{\mathsf{PA}}(p) \quad \Longleftrightarrow \quad \mathcal{K}, \alpha \Vdash p \tag{7.2}$$

For every classical node $\alpha \in K^*$, since the Kripke model above α is just a classical Kripke model, one may repeat the Solovay's argument and show that for every modal proposition B we have

$$\begin{cases} \mathcal{K}, \alpha \Vdash B & \Longrightarrow \quad \mathsf{PA} \vdash L = \alpha \to \sigma_{\mathsf{PA}}(B) \\ \mathcal{K}, \alpha \nVdash B & \Longrightarrow \quad \mathsf{PA} \vdash L = \alpha \to \neg\sigma_{\mathsf{PA}}(B) \end{cases} \tag{7.3}$$

We may use Lemmas 7.18 and 7.19 and eq. (7.2) to conclude

$$\mathcal{K}^*, \alpha \Vdash \sigma_{\mathsf{PA}}(B) \quad \Longleftrightarrow \quad \mathcal{K}, \alpha \Vdash B$$

for every modal proposition B and $\alpha \in K^*$. Since $\mathcal{K}, \alpha_0 \nVdash A$, we have $\mathcal{K}^*, \alpha \nVdash \sigma_{\mathsf{PA}}(A)$, and hence $\mathsf{HA} \nvdash \sigma_{\mathsf{PA}}(A)$, as desired. ❑

Lemma 7.15. *For arbitrary $\alpha, \beta \in K'$ we have*

1. $\mathsf{PA} \vdash \exists x F(x) = \alpha \to \bigvee_{\alpha (\preccurlyeq \cup \sqsubseteq) \beta} L = \beta$,
2. $\mathsf{PA} \vdash L = \alpha \to \neg \Box^+ (L \neq \beta)$, *for every* $\alpha \sqsubset \beta$,
3. $\mathsf{PA} \vdash (L = \alpha) \rhd (L = \beta)$, *for every* $\alpha \prec \beta$,
4. $\mathbb{N} \models L = \alpha_1$,
5. $\mathsf{PA} \vdash L = \alpha \to \Box^+ (L \neq \alpha \wedge \exists x F(x) = \alpha)$, *for every* $\alpha \neq \alpha_1$.

Proof. All proofs are straightforward and left to the reader. ❑

Lemma 7.16. *$\mathcal{I}$, as defined in the proof of Theorem 7.14, is an I-frame (see Definition 3.23).*

Proof. Use Theorem 3.22 and the items 2,3 and 4, of Lemma 7.15. ❑

Lemma 7.17. *For every $\alpha \in K$ we have* $\mathsf{PA} \vdash L = \alpha \to \Box^+ (\bigvee_{\alpha \sqsubset \beta} L = \beta)$.

Proof. It is enough to show that $\mathsf{PA} \vdash L = \alpha \to \Box^+ (L \neq \beta)$ for every $\beta \succcurlyeq \alpha$ such that $\beta \not\sqsupset \alpha$, holds. Consider some $\beta \succcurlyeq \alpha$ with $\beta \not\sqsupset \beta$. If $\beta = \alpha$, by item 5 in Lemma 7.15 we have the desired result. So we may let $\beta \neq \alpha$. We reason inside PA. Let $L = \alpha$. Hence for some x we have $F(x) = \alpha$. Then we reason inside $\Box^+$. By Σ_1-completeness of PA (seeLemma 3.11), we have $F(x) = \alpha$. Assume that $L = \beta$. Let x_0 be the first number such that $F(x_0) = \beta$. Hence for some r such that $\Box^+_r (L \neq \beta)$ holds, we have $F(r) = \alpha_1$. Then $r \leq x$ and hence by Lemma 3.10 we may deduce $L \neq \beta$, in contradiction with $L = \beta$. ❑

Lemma 7.18. *For every α in K and proposition B we have*

$$\mathcal{K}, \alpha \Vdash \Box B \quad \Longrightarrow \quad \mathsf{PA} \vdash L = \alpha \to \sigma_{\mathsf{PA}}(\Box B) \tag{7.4}$$

Proof. Let $\mathcal{K}, \alpha \Vdash \Box B$. Hence for every $\beta \sqsupset \alpha$ we have $\mathcal{K}, \beta \Vdash B$. Since every $\beta \sqsupset \alpha$ is classical, by eq. (7.3) we have $\mathsf{PA} \vdash \bigvee_{\alpha \sqsubset \beta} L = \beta \to \sigma_{\mathsf{PA}}(B)$. Hence $\mathsf{PA} \vdash \Box^+ (\bigvee_{\alpha \sqsubset \beta} L = \beta) \to \sigma_{\mathsf{PA}}(\Box B)$. Lemma 7.17 implies $\mathsf{PA} \vdash L = \alpha \to \sigma_{\mathsf{PA}}(\Box B)$. ❑

Lemma 7.19. *For every α in K and proposition B we have*

$$\mathcal{K}, \alpha \nVdash \Box B \quad \Longrightarrow \quad \mathsf{PA} \vdash L = \alpha \to \neg \sigma_{\mathsf{PA}}(\Box B) \tag{7.5}$$

Proof. Let $\mathcal{K}, \alpha \nVdash \Box B$. Hence for every $\beta \sqsupset \alpha$ we have $\mathcal{K}, \beta \nVdash B$. Since every $\beta \sqsupset \alpha$ is classical, by eq. (7.3) we have $\mathsf{PA} \vdash L = \beta \to \neg \sigma_{\mathsf{PA}}(B)$. Hence $\mathsf{PA} \vdash \sigma_{\mathsf{PA}}(B) \to L \neq \beta$ and then $\mathsf{PA} \vdash \Box^+ \sigma_{\mathsf{PA}}(B) \to \Box^+ L \neq \beta$ and $\mathsf{PA} \vdash \neg \Box^+ L \neq \beta \to \neg \Box^+ \sigma_{\mathsf{PA}}(B)$. Hence item 2 of Lemma 7.15 implies $\mathsf{PA} \vdash L = \alpha \to \neg \Box^+ \sigma_{\mathsf{PA}}(B)$. ❑

7.4 Reductions

Lemma 7.20. *For every $A \in \mathcal{L}_\Box$, if $\mathsf{iH}_\sigma \vdash A^\dagger$ then $\mathsf{iGL\overline{P}C_a} \vdash A$.*

Proof. Let $\mathsf{iGL\overline{P}C_a} \nvdash A$. Since in $\mathsf{iK4} + \Box\mathsf{PEM}$ we have $A \leftrightarrow A^\dagger$, then $\mathsf{iGL\overline{P}C_a} \nvdash A^\dagger$. Hence by $\mathcal{AC}_{\Sigma_1}(\mathsf{iGL\overline{P}C_a}; \mathsf{PA}, \mathsf{PA})$ from Theorem 7.14, we have some Σ_1-substitution σ such that $\mathsf{HA} \nvdash \sigma_{\mathsf{PA}}(A^\dagger)$. Then Lemma 7.8 implies $\mathsf{HA} \nvdash \sigma_{\mathsf{HA}}(A^\dagger)$, and hence by arithmetical soundness of iH_σ (Theorem 5.7) we have $\mathsf{iH}_\sigma \nvdash A^\dagger$, as desired. ❑

Theorem 7.21. $\mathsf{iGL\overline{P}C_a} = \mathcal{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{HA}) \leq \mathcal{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{HA}) = \mathsf{iH}_\sigma$.

Proof. The soundness of $\mathsf{iGL\overline{P}C_a}$ is straightforward and left to the reader. Also by Theorem 5.7, we have $\mathcal{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{HA}) = \mathsf{iH}_\sigma$. So, it is enough to show $\mathcal{AC}_{\Sigma_1}(\mathsf{iGL\overline{P}C_a}; \mathsf{PA}, \mathsf{HA}) \leq_{f,\bar{f}} \mathcal{AC}_{\Sigma_1}(\mathsf{iH}_\sigma; \mathsf{HA}, \mathsf{HA})$. Define $f(A) := A^\dagger$ and $\bar{f}_A$ as identity function.

R1. Use Lemma 7.20.

R2. Use Lemma 7.8. ❑

Theorem 7.22. $\mathsf{GLC_a} = \mathcal{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{PA}) \leq \mathcal{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{PA}) = \mathsf{iH}_\sigma\underline{\mathsf{P}}$.

Proof. We already have $\mathsf{GLC_a} = \mathcal{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{PA})$ and $\mathcal{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{PA}) = \mathsf{iH}_\sigma\underline{\mathsf{P}}$ by Theorems 5.12 and 7.1. So, it is enough to show $\mathcal{AC}_{\Sigma_1}(\mathsf{GLC_a}; \mathsf{PA}, \mathsf{PA}) \leq_{f,\bar{f}} \mathcal{AC}_{\Sigma_1}(\mathsf{iH}_\sigma\underline{\mathsf{P}}; \mathsf{HA}, \mathsf{PA})$.
Define $f(A) := A^\dagger$ and $\bar{f}_A$ as identity function.

R1. Let $\mathsf{iH}_\sigma\underline{\mathsf{P}} \vdash A^\dagger$. By Lemma 7.20 we have $\mathsf{GLC_a} \vdash A$.

R2. Use Lemma 7.8. ❑

The arithmetical completeness of $\mathsf{iGL\overline{P}}$ will be reduced to the one for $\mathsf{iGL\overline{P}C_a}$ via the following lemma. This argument is similar to the one explained in [AM15]. One may use a direct proof for the arithmetical completeness of $\mathsf{iGL\overline{P}}$, similar to what we do for $\mathsf{iGL\overline{P}C_a}$. However this is not enough for our later use insection 8 of the arithmetical completeness of $\mathsf{iGL\overline{P}}$.

Lemma 7.23. *For every modal proposition A, $\mathsf{iGL\overline{P}} \vdash A$ iff for every propositional modal $(.)^{\Box\downarrow}$-substitution τ (Definition 7.2) we have $\mathsf{iGL\overline{P}C_a} \vdash \tau(A)$.*

Proof. One direction holds since $\mathsf{iGL\overline{P}}$ is closed under substitutions and is included in $\mathsf{iGL\overline{P}C_a}$. For the other way around, let $\mathsf{iGL\overline{P}} \nvdash A$. By Theorem 7.12, there is some Suc-classical, semi-perfect $\sqsubset$-branching Kripke model $\mathcal{K} = (K, \preccurlyeq, \sqsubset, V)$ such that $\mathcal{K} \nVdash A$. For every $\alpha \in K$, let p_α be a fresh atomic variable such that for every $\alpha \neq \beta$ we have $p_\alpha \neq p_\beta$. For every $\alpha \in K$, define A_α via induction on the $\prec$-height of α (the maximum number n such that a sequence $\alpha = \alpha_0 \prec \ldots \prec \alpha_n$ exists). So as induction hypothesis, let A_β for every $\beta \succ \alpha$ is defined.

$$A_\alpha^+ := \bigvee_{\alpha \prec \beta} A_\beta \quad , \quad A_\alpha := p_\alpha \wedge \bigwedge_{\alpha \sqsubseteq \beta} \Box\neg \boxdot p_\beta \to A_\alpha^+$$

Let $\bar{\mathcal{K}} = (K, \preccurlyeq, \sqsubset, \bar{V})$, in which $\alpha \bar{V} p$ iff $p = p_\beta$ for some $\beta(\preccurlyeq \cup \sqsubset)\alpha$. Define

$$\tau(p) := \bigvee_{\mathcal{K}, \alpha \Vdash p} A_\alpha$$

Then by induction on the complexity of the modal proposition B, we show

$$\mathcal{K}, \alpha \Vdash B \iff \bar{\mathcal{K}}, \alpha \Vdash \tau(B)$$

- B is atomic variable: For every $\alpha \in K$ such that $\mathcal{K}, \alpha \Vdash B$, by Lemma 7.24 we have $\bar{\mathcal{K}}, \alpha \Vdash A_\alpha$ and hence $\bar{\mathcal{K}}, \alpha \Vdash \tau(p)$. Also if $\bar{\mathcal{K}}, \alpha \Vdash \tau(B)$, then for some $\beta \in K$ we have $\mathcal{K}, \beta \Vdash B$ and $\bar{K}, \alpha \Vdash A_\beta$. Hence by Lemma 7.24 we have $\beta \preccurlyeq \alpha$, which implies $\mathcal{K}, \alpha \Vdash B$, as desired.
- All the other cases are trivial and left to the reader.

Then we have $\bar{\mathcal{K}} \nVdash \tau(A)$. Obviously the Kripke model $\bar{\mathcal{K}}$ inherits all properties from $\mathcal{K}$ and moreover it is atom-complete. Hence by soundness part of the Theorem 7.13, $\mathsf{iGL\overline{P}C_a} \nvdash \tau(A)$, as desired. ❑

Lemma 7.24. *Let $\bar{\mathcal{K}}$ and A_α, as defined in the proof of Lemma 7.23. For every $\alpha, \beta \in K$ we have $\bar{\mathcal{K}}, \alpha \Vdash A_\beta$ iff $\alpha \succcurlyeq \beta$.*

Proof. We use induction on the $\prec$-height of β. As induction hypothesis, let for every $\beta \succ \beta_0$ and $\alpha \in K$ we have $\bar{\mathcal{K}}, \alpha \Vdash A_\beta$ iff $\beta \succcurlyeq \alpha$. Note that by induction hypothesis we have $\bar{\mathcal{K}}, \beta \Vdash A^+_{\beta_0}$ iff $\beta \succ \beta_0$.

- ($\alpha \succcurlyeq \beta_0$ implies $\bar{\mathcal{K}}, \alpha \Vdash A_{\beta_0}$): It is enough to show that $\bar{\mathcal{K}}, \beta_0 \Vdash A_{\beta_0}$. Then for evey $\alpha \succcurlyeq \beta_0$ we have $\bar{\mathcal{K}}, \alpha \Vdash A_{\beta_0}$, as desired. By definition of $\bar{\mathcal{K}}$, we have $\bar{\mathcal{K}}, \beta_0 \Vdash p_{\beta_0}$. Consider some $\gamma \sqsupseteq \beta_0$. Again by definition of $\bar{\mathcal{K}}$, we have $\bar{\mathcal{K}}, \beta_0 \nVdash \Box\neg p_\gamma$ and for every $\delta \succ \beta_0$ we have $\bar{\mathcal{K}}, \delta \Vdash A^+_{\beta_0}$. Hence $\bar{\mathcal{K}}, \beta_0 \Vdash \Box\neg p_\gamma \to A^+_{\beta_0}$. This argument shows that $\bar{\mathcal{K}}, \beta_0 \Vdash A_{\beta_0}$, as desired.
- ($\bar{\mathcal{K}}, \alpha \Vdash A_{\beta_0}$ implies $\alpha \succcurlyeq \beta_0$): Let $\bar{\mathcal{K}}, \alpha \Vdash A_{\beta_0}$. Since $\bar{\mathcal{K}}, \alpha \Vdash p_{\beta_0}$, we have $\beta_0(\preccurlyeq \cup \sqsubset)\alpha$. If $\beta_0 \preccurlyeq \alpha$, we are done. So let $\beta_0 \npreccurlyeq \alpha$ and $\beta_0 \sqsubset \alpha$. Hence for arbitrary $\gamma \sqsupseteq \beta_0$ we have $\bar{\mathcal{K}}, \alpha \Vdash \neg\Box\neg p_\gamma$. This by Suc-classicality, implies that there is some $\delta \sqsupset \alpha$ such that $\bar{\mathcal{K}}, \delta \Vdash p_\gamma$. Then we have $\gamma(\preccurlyeq \cup \sqsubset)\delta$. By Suc-classicality, we have $\gamma \sqsubseteq \delta$. Since $\bar{\mathcal{K}}$ is with tree frame, we have either $\alpha \sqsubseteq \gamma$ or $\gamma \sqsubseteq \alpha$. On the other hand, since $\bar{\mathcal{K}}$ is $\sqsubset$-branching, there must be some $\gamma \sqsupset \beta_0$ which is $\sqsubset$-incomparable with α, a contradiction with our previous argument. ❑

Theorem 7.25. $\mathsf{iGL\overline{P}} = \mathcal{PL}(\mathsf{PA}, \mathsf{HA}) \leq \mathcal{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{HA}) = \mathsf{iGL\overline{P}C_a}$.

Proof. The arithmetical soundness of $\mathsf{iGL\overline{P}}$ is straightforward and left to the reader. Also by Theorem 7.21 we have $\mathcal{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{HA}) = \mathsf{iGL\overline{P}C_a}$. It remains to show

$$\mathcal{AC}(\mathsf{iGL\overline{P}}; \mathsf{PA}, \mathsf{HA}) \leq_{f,\bar{f}} \mathcal{AC}_{\Sigma_1}(\mathsf{iGL\overline{P}}; \mathsf{PA}, \mathsf{HA})$$

Let $A \in \mathcal{L}_\Box$ such that $\mathsf{iGL\overline{P}} \nvdash A$. Then by Lemma 7.23 there is some substitution τ such that $\mathsf{iGL\overline{P}C_a} \nvdash \tau(A)$. Define the function f as follows:

$$f(A) := \begin{cases} \tau(A) & : \mathsf{iGL\overline{P}} \nvdash A \\ \text{whatever you like} & : \text{otherwise} \end{cases}$$

Also let $\bar{f}_A(\sigma) := \sigma_{\mathsf{PA}} \circ \tau$. Then one may easily observe that R0, R1 and R3 holds for this $f, \bar{f}$. ❑

Lemma 7.26. *For $A \in \mathcal{L}_\Box$, if $\mathsf{iGL\overline{P}C_a} \vdash A^{\neg\uparrow}$ then $\mathsf{GLC_a} \vdash A$.*

Proof. Let $\mathsf{iGL\overline{P}C_a} \vdash A^{\neg\uparrow}$. Then $\mathsf{GLC_a} \vdash A^{\neg\uparrow}$ and since $A^{\neg\uparrow}$ is classically equivalent to A we have $\mathsf{GLC_a} \vdash A$. ❑

Theorem 7.27. $\mathsf{GLC_a} = \mathcal{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{PA}) \leq \mathcal{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{HA}) = \mathsf{iGL\overline{P}C_a}$.

Proof. By Theorems 7.1 and 7.21 we have $\mathsf{GLC_a} = \mathcal{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{PA})$ and $\mathcal{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{HA}) = \mathsf{iGL\overline{P}C_a}$. We must show $\mathcal{AC}_{\Sigma_1}(\mathsf{GLC_a}; \mathsf{PA}, \mathsf{PA}) \leq_{f,\bar{f}} \mathcal{AC}_{\Sigma_1}(\mathsf{iGL\overline{P}C_a}; \mathsf{PA}, \mathsf{HA})$. Given $A \in \mathcal{L}_\Box$, define $f(A) := A^{\neg\uparrow}$ and $\bar{f}_A$ as identity function.

R1. If $\mathsf{iGL\overline{P}C_a} \vdash A^{\neg\uparrow}$, then by Lemma 7.26 we have $\mathsf{GLC_a} \vdash A$.

R2. Holds by Lemmas 5.15 and 5.18. ❑

Theorem 7.28. $\mathsf{iGL\overline{P}C_a} = \mathcal{PL}_{\Sigma_1}(\mathsf{PA},\mathsf{HA}) \leq \mathcal{PL}_{\Sigma_1}(\mathsf{PA},\mathbb{N}) = \mathsf{GL\underline{S}C_a}$.

Proof. By Theorems 7.1 and 7.25 we have $\mathcal{PL}_{\Sigma_1}(\mathsf{PA},\mathbb{N}) = \mathsf{GL\underline{S}C_a}$ and $\mathsf{iGL\overline{P}C_a} = \mathcal{PL}_{\Sigma_1}(\mathsf{PA},\mathsf{HA})$. We must show $\mathcal{AC}_{\Sigma_1}(\mathsf{iGL\overline{P}};\mathsf{PA},\mathsf{HA}) \leq_{f,\bar{f}} \mathcal{AC}_{\Sigma_1}(\mathsf{GL\underline{S}C_a};\mathsf{PA},\mathbb{N})$. Given $A \in \mathcal{L}_\Box$, define $f(A) = \Box A$ and $\bar{f}_A$ as identity function.

R1. Let $\mathsf{GL\underline{S}C_a} \vdash \Box A$. By soundness of $\mathsf{iH_\sigma \underline{SP}} = \mathcal{PL}_{\Sigma_1}(\mathsf{HA},\mathbb{N})$, for every Σ_1-substitution σ we have $\mathbb{N} \models \sigma_{\mathsf{HA}}(\Box A)$ and hence $\mathsf{HA} \vdash \sigma_{\mathsf{HA}}(A)$. Then by arithmetical completeness of $\mathcal{PL}_{\Sigma_1}(\mathsf{HA},\mathsf{HA})$, we have $\mathsf{iH_\sigma} \vdash A$.
One also may prove this item with a direct propositional argument. For simplicity reasons, we chose the indirect way.

R2. Let $\mathbb{N} \not\models \sigma_{\mathsf{HA}}(\Box A)$. Then $\mathsf{HA} \nvdash \sigma_{\mathsf{HA}}(A)$, as desired. ❑

Theorem 7.29. $\mathsf{GLC_a} = \mathcal{PL}_{\Sigma_1}(\mathsf{PA},\mathsf{PA}) \leq \mathcal{PL}_{\Sigma_1}(\mathsf{PA},\mathbb{N}) = \mathsf{GL\underline{S}C_a}$.

Proof. By Theorem 7.1 we have $\mathsf{GLC_a} = \mathcal{PL}_{\Sigma_1}(\mathsf{PA},\mathsf{PA})$ and $\mathcal{PL}_{\Sigma_1}(\mathsf{PA},\mathbb{N}) = \mathsf{GL\underline{S}C_a}$. We must show $\mathcal{AC}_{\Sigma_1}(\mathsf{GLC_a};\mathsf{PA},\mathsf{PA}) \leq_{f,\bar{f}} \mathcal{AC}_{\Sigma_1}(\mathsf{GL\underline{S}C_a};\mathsf{PA},\mathbb{N})$. Given $A \in \mathcal{L}_\Box$, define $f(A) = \Box A$ and $\bar{f}_A$ as identity function.

R1. Let $\mathsf{GL\underline{S}C_a} \vdash \Box A$. By soundness of $\mathsf{GL\underline{S}C_a} = \mathcal{PL}_{\Sigma_1}(\mathsf{PA},\mathbb{N})$, for every Σ_1-substitution σ we have $\mathbb{N} \models \sigma_{\mathsf{PA}}(\Box A)$ and hence $\mathsf{PA} \vdash \sigma_{\mathsf{PA}}(A)$. Then by arithmetical completeness of $\mathcal{PL}_{\Sigma_1}(\mathsf{PA},\mathsf{PA})$, we have $\mathsf{GLC_a} \vdash A$.
One also may prove this item with a direct propositional argument, using Kripke semantics. For simplicity reasons, we chose the indirect way.

R2. Let $\mathbb{N} \not\models \sigma_{\mathsf{PA}}(\Box A)$. Then $\mathsf{PA} \nvdash \sigma_{\mathsf{PA}}(A)$, as desired. ❑

Theorem 7.30. $\mathsf{GL} = \mathcal{PL}_{\Sigma_1}(\mathsf{PA},\mathsf{PA}) \leq \mathcal{PL}_{\Sigma_1}(\mathsf{PA},\mathbb{N}) = \mathsf{GL\underline{S}}$.

Proof. Similar to the proof of Theorem 7.29 and left to the reader. ❑

8 Relative provability logics for PA^*

In this section, we characterize several relative provability logics for PA^* via reductions. All reductions are shown at once in the diagram 4. The head of arrow reduces to its tail, via some simple reduction (section 4.1). The translation f in the reduction, is shown over the arrow lines and the number which appears under arrow, is the corresponding theorem.

8.1 Kripke Semantics

In the following lemma, we will show that the axioms CP and TP are local over iGL, i.e. whenever we can deduce some proposition A from $\mathsf{CP}+\mathsf{TP}$ in iGL, then we may deduce it by those instances of CP and TP which use the subformulas of A:

Lemma 8.1. *For every A, if $\mathsf{iGLCT} \vdash A$ then*

$$\mathsf{iGL} \vdash \boxdot \left[\bigwedge_{E\to F\in \mathsf{sub}(A)} \Box(E\to F) \to (E\vee(E\to F)) \wedge \bigwedge_{E\in\mathsf{sub}(A)} (E\to\Box E)\right] \to A$$

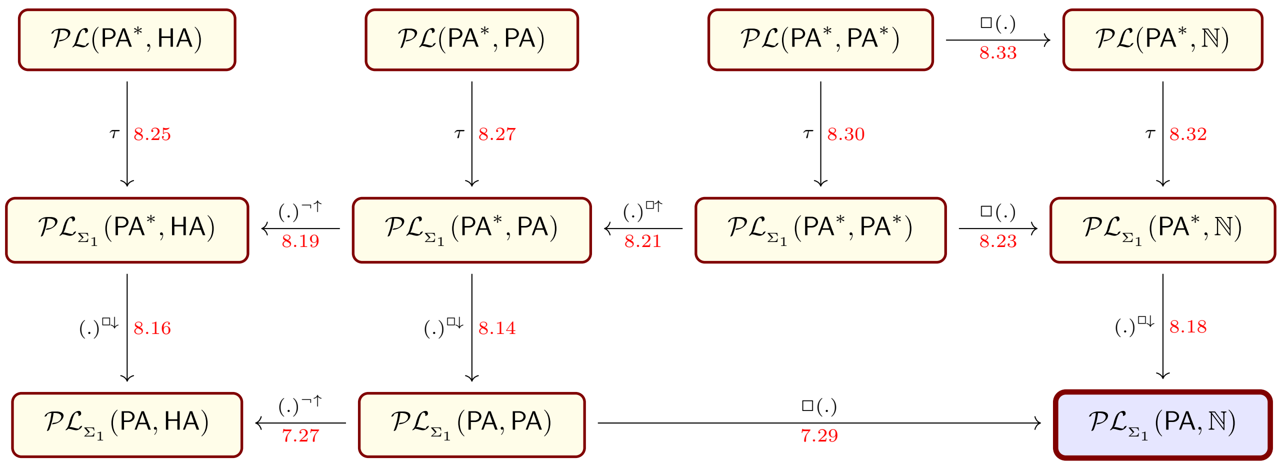

Diagram 4: Reductions for relative provability logics of PA^*

Proof. For the simplicity of notations, in this proof, let

$$\varphi := \boxdot \bigwedge_{E\to F\in\mathsf{sub}(A)} \Box(E\to F)\to(E\vee(E\to F))\wedge\boxdot\bigwedge_{E\in\mathsf{sub}(A)}(E\to\Box E)$$

and $\vdash$ indicates derivablity in $\mathsf{iGL}+\varphi$.
One side is trivial. For the other way around, assume that $\mathsf{iGL}\nvdash\varphi\to A$. We will construct some finite Kripke model $\mathcal{K}=(K,\sqsubset,\preccurlyeq,V)$ with $\sqsubset=\prec$ such that $\mathcal{K},\alpha\nVdash A$, which by soundness of iGLCT for finite Kripke models with $\prec=\sqsubset$, we have the desired result. The proof is almost identical to the proof of Theorem 3.38 in [AM18, Theorem 4.26]. To be self-contained, we elaborate it here.

Let $\mathsf{Sub}(A)$ be the set of sub-formulae of A. Then define

$$X := \{B, \Box B \mid B\in\mathsf{Sub}(A)\}$$

It is obvious that X is a finite adequate set. We define $\mathcal{K}=(K,\preccurlyeq,\sqsubset,V)$ as follows. Define

- K as the set of all X-saturated sets with respect to $\mathsf{iGL}+\varphi$,
- $\alpha\sqsubset\beta$ iff $\{D:\Box D\in\alpha\}\subseteq\beta$ and $\alpha\subsetneqq\beta$,
- $\alpha\preccurlyeq\beta$ iff $\alpha\sqsubset\beta$ or $\alpha=\beta$,
- $\alpha V p$ iff $p\in\alpha$, for atomic p.

$\mathcal{K}$ trivially satisfies all the properties of finite Kripke model with $\sqsubset=\prec$. So we must only show that $\mathcal{K}\nVdash A$. To this end, we first show by induction on $B\in X$ that $B\in\alpha$ iff $\alpha\Vdash B$, for each $\alpha\in K$. The only non-trivial cases are $B=\Box C$ and $B=E\to F$.

- $B=\Box C$: Let $\Box C\notin\alpha$. We must show $\alpha\nVdash\Box C$. The other direction is easier to prove and we leave it to reader. Let $\beta_0:=\{D\in X\mid\alpha\vdash\Box D\}$. If $\beta_0,\Box C\vdash C$, since by definition of β_0, we have $\alpha\vdash\Box\beta_0$ and hence by Löb's axiom, $\alpha\vdash\Box C$, which is in contradiction with $\Box C\notin\alpha$. Hence $\beta_0,\Box C\nvdash C$ and so there exists some X-saturated set β such that $\beta\nvdash C$, $\beta\supseteq\beta_0\cup\{\Box C\}$. Hence $\beta\in K$ and $\alpha\sqsubset\beta$. Then by the induction hypothesis, $\beta\nVdash C$ and hence $\alpha\nVdash\Box C$.

- Let $E \to F \notin \alpha$. Then $F \notin \alpha$. If $E \in \alpha$, by induction hypothesis we have $\alpha \Vdash E$ and $\alpha \nVdash F$, and hence $\alpha \nVdash E \to F$, as desired. So we may let $E \notin \alpha$. Define $\beta_0 := \{D : \Box D \in \alpha\}$. If $\alpha \vdash \bigwedge \beta_0 \to (E \to F)$, then $\alpha \vdash \Box(E \to F)$ and hence by TP, either we have $\alpha \vdash E$ or $\alpha \vdash E \to F$, a contradiction. So we may let $\alpha \nvdash \bigwedge \beta_0 \to (E \to F)$, and use Lemma 3.37 to find $\beta \supseteq \beta_0 \cup \alpha \cup \{E\}$ as some X-saturated node in $\mathcal{K}$. Hence $\alpha \sqsubset \beta$ which implies $\alpha \prec \beta$ and by induction hypothesis $\beta \Vdash E$ and $\beta \nVdash F$, which implies $\alpha \nVdash E \to F$, as desired.

Since $\mathsf{iGL} + \varphi \nvdash A$, by Lemma 3.37, there exists some X-saturated set $\alpha \in K$ such that $\alpha \nvdash A$, and hence by the above argument we have $\alpha \nVdash A$. ❑

Lemma 8.2. *For arbitrary proposition A*

$$\mathsf{iGLCT} \vdash A^{\Box} \textit{ implies } \mathsf{iGL} + \Box\mathsf{CP} + \mathsf{TP} \vdash A^{\Box}.$$

Proof. Let $\mathsf{iGLCT} \vdash A^{\Box}$. Hence by Lemma 8.1 the following is derivable in iGL

$$\left[\boxdot \overbrace{\bigwedge_{B \in \mathsf{sub}(A^{\Box})} B \to \Box B}^{G} \wedge \boxdot \overbrace{\bigwedge_{E \to F \in \mathsf{sub}(A^{\Box})} \Box(E \to F) \to (E \vee (E \to F))}^{H} \right] \to A^{\Box}$$

Hence by Lemma 3.7 $\mathsf{iGL} \vdash G^{\Box\uparrow} \wedge H^{\Box\uparrow} \to A^{\Box}$. By Lemma 3.8, $\mathsf{iK4} \vdash G^{\Box\uparrow}$. Also $(\Box G)^{\Box\uparrow} = \Box G$ which is an instance of $\Box\mathsf{CP}$. Let us consider some arbitrary conjunct $\Box(E \to F) \to (E \vee (E \to F))$ in H. Since $E \to F$ is a subformula of $A^{\Box}$, we have $E = E_0^{\Box}$ and $F = F_0^{\Box}$. Hence inside $\mathsf{iK4}$, the $H^{\Box\uparrow}$ is equivalent to some instance of TP. Hence $\mathsf{iGL} + \Box\mathsf{CP} + \mathsf{TP} \vdash A^{\Box}$. ❑

Theorem 8.3. $\mathsf{iGL\overline{CT}}$ *is sound and complete for semi-perfect* Suc*-quasi-classical Kripke models.*

Proof. The soundness is easy and left to the reader. For the completeness, we first show the completeness for finite brilliant irreflexive transitive Suc-quasi-classical Kripke models. Let $\mathsf{iGL\overline{CT}} \nvdash A$. Let

$$X := \{B, \Box B : B \in \mathsf{Sub}(A)\}$$

and define the Kripke model $\mathcal{K} = (K, \preccurlyeq, \sqsubset, V)$ as follows:

- K is the family of all X-saturated sets with respect to $\mathsf{iGL\overline{CT}}$.
- $\alpha \sqsubset \beta$ iff $\alpha \neq \beta$ and $\{B, \Box B : \Box B \in \alpha\} \subseteq \beta$ and β is X-saturated with respect to iGLCT.
- $\alpha \prec \beta$ iff $\alpha \subsetneqq \beta$ and either $\alpha \sqsubset \beta$ or $\gamma \not\subseteq \alpha$, for every $\gamma \in K$.
- $\alpha \, V \, p$ iff $p \in \alpha$.

It is straightforward to show that $\mathcal{K}$ is a finite brilliant irreflexive transitive Suc-quasi-classical Kripke model. We leave them to the reader. We only show that $\mathcal{K}, \alpha \Vdash B$ iff $B \in \alpha$ for every $\alpha \in K$ and $B \in X$. Then by Lemma 3.37 one may find some $\alpha \in K$ such that $\mathcal{K}, \alpha \nVdash A$, as desired.

Use induction on the complexity of $B \in X$. All inductive steps are trivial, except for:

- $B = \Box C$: If $\Box C \in \alpha$ and $\alpha \sqsubset \beta$, then by definition, $C \in \beta$ and hence by induction hypothesis $\beta \Vdash C$. This implies $\alpha \Vdash \Box C$. For the other way around, let $\Box C \notin \alpha$. Consider the set $\Delta := \{E, \Box E : \Box E \in \alpha\}$. If $\mathsf{iGLCT} \vdash \bigwedge \Delta \to (\Box C \to C)$, then $\mathsf{iGL\overline{CT}} \vdash \Box(\bigwedge \Delta) \to \Box C$. Since $\mathsf{iK4}{+}\alpha \vdash \Box(\bigwedge \Delta)$, we may deduce $\mathsf{iGL\overline{CT}}{+}\alpha \vdash \Box C$, a contradiction. Hence $\mathsf{iGLCT} \nvdash (\bigwedge \Delta \wedge \Box C) \to C$. By Lemma 3.37, there exists some X-saturated set $\beta \supseteq \Delta \cup \{\Box C\}$ with respect to iGLCT such that $C \notin \beta$. Hence $\beta \in K$ and $\alpha \sqsubset \beta$ and $C \notin \beta$. Induction hypothesis implies that $\beta \nVdash C$ and hence $\alpha \nVdash \Box C$.
- $B = C \to D$: If $C \to D \in \alpha$ and $\alpha \preccurlyeq \beta$ and $\beta \Vdash C$, by induction hypothesis $C \in \beta$ and hence $D \in \beta$. Again by induction hypothesis we have $\beta \Vdash D$. This shows that $\alpha \Vdash C \to D$. For the other way around, let $C \to D \notin \alpha$. We have two cases:

- There is some $\gamma \sqsubset \alpha$: Hence α is X-sturated w.r.t iGLCT. Let $\Delta := \{E : \Box E \in \alpha\}$. We have tow subcases:
  - If $\mathsf{iGLCT} + \Delta + \alpha \vdash C \to D$, then $\mathsf{iGLCT} + \Box\alpha + \Box\Delta \vdash \Box(C \to D)$. By the completeness principle, we have $\mathsf{iGLCT} + \alpha \vdash \Box(C \to D)$. By TP we have $\mathsf{iGLCT} + \alpha \vdash C \vee (C \to D)$. Since α is X-saturated with respect to iGLCT, we have either $C \in \alpha$ or $C \to D \in \alpha$. The latter is impossible, hence $C \in \alpha$. Again by X-saturatedness of α, we can deduce $D \notin \alpha$. Hence by induction hypothesis we have $\alpha \Vdash C$ and $\alpha \nVdash D$, which implies $\alpha \nVdash C \to D$, as desired.
  - If $\mathsf{iGLCT}+\Delta+\alpha \nvdash C \to D$, then by Lemma 3.37 there exists some X-saturated $\beta \supseteq \alpha \cup \Delta \cup \{C\}$ w.r.t iGLCT (and a fortiori $\mathsf{iGL\overline{CT}}$) such that $D \notin \beta$. Induction hypothesis implies $\beta \Vdash C$ and $\beta \nVdash D$. One may observe that $\alpha \prec \beta$ or $\alpha = \beta$, and hence $\alpha \nVdash C \to D$, as desired.
- There is no $\gamma \sqsubset \alpha$: since $\mathsf{iGL\overline{CT}} + \alpha \nvdash C \to D$, by Lemma 3.37, there exists some X-saturate set $\beta \supseteq \alpha \cup \{C\}$ with respect to $\mathsf{iGL\overline{CT}}$ such that $D \notin \beta$. Hence by induction hypothesis $\beta \Vdash C$ and $\beta \nVdash D$. One may observe that $\beta \succcurlyeq \alpha$ and hence $\alpha \nVdash C \to D$.

Next we use the construction method [Iem01] to fulfil the other conditions: being neat and tree. Let $\mathcal{K}_t := (K_t, \preccurlyeq_t, \sqsubset_t, V_t)$ as follows:

- K_t is the set of all finite sequences $r := \langle \alpha_0, \ldots \alpha_n \rangle$ such that for any $i < n$ either we have $\alpha_i \prec \alpha_{i+1}$ or $\alpha_i \sqsubset \alpha_{i+1}$. Let $f(r)$ indicates the final element of the sequence r.
- $r \preccurlyeq_t s$ iff r is an initial segment of s and $f(r) \preccurlyeq f(s)$.
- $r \sqsubset_t s$ iff r is an initial segment of $s = \langle \alpha_0, \ldots \alpha_n \rangle$, e.g. $r = \langle \alpha_0, \ldots \alpha_k \rangle$ for some $k < n$ and $\alpha_i \sqsubset \alpha_{i+1}$ for some $k \leq i < n$.
- $r\, V_t\, p$ iff $f(r)\, V\, p$.

It is straightforward to show that $\mathcal{K}_t$ is semi-perfect Suc-quasi-classical Kripke model and for every $r \in K_t$ and formula B we have

$$\mathcal{K}_t, r \Vdash B \qquad \Longleftrightarrow \qquad \mathcal{K}, f(r) \Vdash B.$$

❑

Theorem 8.4. $\mathsf{iGL\overline{CT}C_a}$ *is sound and complete for semi-perfect* Suc*-quasi-classical atom-complete Kripke models.*

Proof. Similar to the proof of Theorem 8.3 and left to the reader. ❑

Theorem 8.5. *For every proposition A, we have* $\mathsf{iGL\overline{C}T\underline{P}} \vdash A$ *iff for every quasi-classical perfect Kripke model $\mathcal{K}$ and every boolean interpretation I and arbitrary node α in $\mathcal{K}$ we have $\mathcal{K}, \alpha, I \models A$.*

Proof. The soundness is easy and left to the reader. For the completeness part, let $\mathsf{iGL\overline{C}T\underline{P}} \nvdash A$. Let A' be a boolean equivalent of A which is a conjunction of implications $E \to F$ in which E is a conjunction of a set of atomics or boxed propositions and F is a disjunction of atomics or boxed proposition. Evidently such A' exists for every A. Hence $\mathsf{iGL\overline{C}T\underline{P}} \nvdash A'$. Then there must be some conjunct $E \to F$ of A' such that $\mathsf{iGL\overline{C}T\underline{P}} \nvdash E \to F$, E is a conjunction of atomic and boxed propositions and F is a disjunction of atomic and boxed propositions. Let X_E be the set of atomic conjuncts in E and X_F the set of atomic disjuncts in F. Note here that X_E and X_F are disjoint sets. Define $\bar{E}$ and $\bar{F}$ as the replacement of X_E and X_F by $\top$ and $\bot$ in E and F, respectively. Hence $\bar{E} \to \bar{F}$, does not have any outer atomics and then $(\bar{E} \to \bar{F})^{\Box}$ is equivalent in $\mathsf{iGL} + \Box\mathsf{CP}$ with $\bar{E} \to \bar{F}$. Then $\mathsf{iGL} + \mathsf{TP} + \Box\mathsf{CP} \nvdash (E \to F)^{\Box}$ and by Lemma 8.2 we have $\mathsf{iGLCT} \nvdash \bar{E} \to \bar{F}$. Then by Theorem 3.38, there is some perfect, quasi-classical Kripke model $\mathcal{K}$ such that $\mathcal{K}, \alpha \nVdash \bar{E} \to \bar{F}$. Hence there is some $\beta \succcurlyeq \alpha$ such that $\mathcal{K}, \beta \Vdash \bar{E}$ and $\mathcal{K}, \beta \nVdash \bar{F}$. Let the boolean interpretation I defined such that:

$$I(p) := \begin{cases} \text{true} & : p \in X_E \\ \text{false} & : p \in X_F \\ \text{no matter, true or false} & : \text{otherwise} \end{cases}$$

One may observe that $\mathcal{K}, \beta, I \not\models E \to F$ and hence $\mathcal{K}, \beta, I \not\models A$. ❑

Theorem 8.6. *For every proposition A, we have $\mathsf{iGL\overline{C}T\underline{P}C_a} \vdash A$ iff for every quasi-classical perfect Kripke model $\mathcal{K}$ and arbitrary node α in $\mathcal{K}$ we have $\mathcal{K}, \alpha \models A$.*

Proof. The proof is very similar to the one for Theorem 8.5, except for the argument for X_E and X_F and $\bar{E}$ and $\bar{F}$ and the boolean interpretation I, which are unnecessary here with the presence of the $\mathsf{CP_a}$. For readability reasons, we bring the adapted proof here.

The soundness is straightforward and left to the reader. For the completeness, let $\mathsf{iGL\overline{C}T\underline{P}C_a} \nvdash A$. Let A' be a boolean equivalent of A which is a conjunction of implications $E \to F$ in which E is a conjunction of a set of atomics or boxed propositions and F is a disjunction of atomics or boxed proposition. Evidently such A' exists for every A. Hence $\mathsf{iGL\overline{C}T\underline{P}C_a} \nvdash A'$. Then there must be some conjunct $E \to F$ of A' such that $\mathsf{iGL\overline{C}T\underline{P}} \nvdash E \to F$, E is a conjunction of atomic and boxed propositions and F is a disjunction of atomic and boxed propositions. Hence $E^\Box \to F^\Box$ is equivalent in $\mathsf{iK4} + \mathsf{CP_a} + \Box\mathsf{CP}$ with $E \to F$. Then $\mathsf{iGL} + \mathsf{TP} + \Box\mathsf{CP} + \mathsf{CP_a} \nvdash (E \to F)^\Box$ and by Lemma 8.2 we have $\mathsf{iGLCT} \nvdash E \to F$. Then by Theorem 3.38, there is some perfect, quasi-classical Kripke model $\mathcal{K}$ such that $\mathcal{K}, \alpha \nVdash E \to F$. Hence there is some $\beta \succcurlyeq \alpha$ such that $\mathcal{K}, \beta \Vdash E$ and $\mathcal{K}, \beta \nVdash F$. Then $\mathcal{K}, \beta \not\models E \to F$ and hence $\mathcal{K}, \beta \not\models A$. ❑

Theorem 8.7. $\mathsf{iGL\overline{C}T\underline{S^*P}} \vdash A^{\Box\downarrow}$ *iff for every quasi-classical perfect Kripke model $\mathcal{K}$ and every boolean interpretation I and arbitrary $A^{\Box\downarrow}$-sound node α in $\mathcal{K}$ we have $\mathcal{K}, \alpha, I \models A^{\Box\downarrow}$.*

Proof. Both directions are proved contra-positively. For the soundness part, assume that $\mathcal{K}, \alpha, I \not\models A^{\Box\downarrow}$ for some boolean interpretation I and quasi-classical perfect Kripke model $\mathcal{K} := (K, \preccurlyeq, \sqsubset, V)$ which is $A^{\Box\downarrow}$-sound at $\alpha \in K$. Since derivability is finite, it is enough to show that for every finite set Γ of modal propositions we have

$$\mathsf{iGL\overline{C}T\underline{P}} \nvdash \bigwedge_{B \in \Gamma} (\Box B^\Box \to B^\Box) \to A^{\Box\downarrow}.$$

By Theorem 8.5 and Lemma 3.43, it is enough to find some number i such that

$$\mathcal{K}^{(i)}, \alpha_i, I \not\models \bigwedge_{B \in \Gamma} (\Box B^\Box \to B^\Box) \to A^{\Box\downarrow}.$$

Let us define n_i and m_i as the number of propositions in the sets $N_i := \{B \in \Gamma : \mathcal{K}^{(i)}, \alpha_i, I \models B^\Box \wedge \Box B^\Box\}$ and $M_i := \{B \in \Gamma : \mathcal{K}^{(i)}, \alpha_i, I \models \Box B^\Box \wedge \neg B^\Box\}$, respectively. We use induction on k and prove the following statement:

$$\varphi(k) := \text{for every } i, \text{ if } n_i < k \text{ then there is some } 0 \leq j \leq 1 + n_i \text{ such that}$$

$$\mathcal{K}^{(i+j)}, \alpha_{i+j}, I \models \bigwedge_{B \in \Gamma} (\Box B^\Box \to B^\Box) \tag{8.1}$$

Then by $\varphi(n_0 + 1)$, one may find some number j such that $\mathcal{K}^j, \alpha_j, I \models \bigwedge_{B \in \Gamma}(\Box B^\Box \to B^\Box)$, and by Lemma 3.43 we also have $\mathcal{K}^j, \alpha_j, I \not\models A^{\Box\downarrow}$, as desired.

$\varphi(0)$ trivially holds. As induction hypothesis, let $\varphi(k)$ holds and show that $\varphi(k+1)$ holds as follows. Let some number i such that $n_i < k + 1$. If $n_i < k$, by induction hypothesis we have the desired conclusion. So let $n_i = k$. If $m_i = 0$, we may let $j = 0$ and we have eq. (8.1). So let $B \in \Gamma$ such that $\mathcal{K}^{(i)}, \alpha_i, I \models \Box B^\Box \wedge \neg B^\Box$. We have two sub-cases:

- $m_{i+1} = 0$: observe in this case that eq. (8.1) holds for $j = 1$.
- $m_{i+1} > 0$: in this case we have $n_{i+1} < k$ and hence by application of the induction hypothesis with $i := i + 1$, we get some $0 \leq j' \leq 1 + n_{i+1}$ such that $\mathcal{K}^{(i+1+j')}, \alpha_{i+1+j'} \models \bigwedge_{B \in \Gamma}(\Box B^\Box \to B^\Box)$. Hence if we let $j := j' + 1$ we have $0 \leq j \leq 1 + n_i$ and eq. (8.1), as desired.

For the completeness part, assume that $\mathsf{iGL\overline{C}T\underline{S^*P}} \nvdash A^{\Box\downarrow}$. Hence

$$\mathsf{iGL\overline{C}T\underline{P}} \nvdash \left(\bigwedge_{\Box B^{\Box} \in \mathsf{Sub}(A^{\Box\downarrow})} (\Box B^{\Box} \to B^{\Box}) \right) \to A^{\Box\downarrow}$$

Hence Theorem 8.5 implies the desired result. ❑

Theorem 8.8. $\mathsf{iGL\overline{C}T\underline{S^*P}C_a} \vdash A^{\Box\downarrow}$ *iff for every quasi-classical perfect Kripke model* $\mathcal{K}$ *and arbitrary* A*-sound node* α *in* $\mathcal{K}$ *we have* $\mathcal{K}, \alpha \models A$.

Proof. The proof is similar to the one for Theorem 8.7. One must use Theorem 8.6 instead of Theorem 8.5 in the proof. ❑

8.2 Reductions

Lemma 8.9. $\mathsf{iGLCT} \vdash A$ *implies* $\mathsf{GL} \vdash A^{\Box}$.

Proof. Use induction on the proof $\mathsf{iGLCT} \vdash A$. ❑

Lemma 8.10. $\mathsf{GL} \vdash A$ *implies* $\mathsf{iGL\overline{P}} \vdash \Box A$.

Proof. Let $\mathsf{GL} \vdash A$. Hence $\mathsf{iGL} \vdash \boxdot \mathsf{PEM} \to A$. Since necessitation is admissible to iGL, we have $\mathsf{iGL} \vdash \Box \mathsf{PEM} \to \Box A$ which implies $\mathsf{iGL\overline{P}} \vdash \Box A$. ❑

Definition 8.11. For a Kripke model $\mathcal{K} = (K, \preccurlyeq, \sqsubset, V)$, let $\tilde{\mathcal{K}}$, indicates the Kripke model derived from $\mathcal{K}$ by making every $\sqsubset$-accessible node as a classical node. More precisely, we define $\tilde{\mathcal{K}} := (K, \tilde{\preccurlyeq}, \sqsubset, V))$ in this way:

$$\alpha \tilde{\preccurlyeq} \beta \text{ iff “}\alpha \text{ is not } \sqsubset\text{-accessible } (\alpha \notin \mathsf{Suc}) \text{ and } \alpha \preccurlyeq \beta\text{” or}$$
$$\text{“}\alpha \text{ is } \sqsubset\text{-accessible } (\alpha \notin \mathsf{Suc}) \text{ and } \alpha = \beta\text{”}$$

Lemma 8.12. *For every* Suc*-quasi-classical semi-perfect Kripke model* $\mathcal{K} = (K, \preccurlyeq, \sqsubset, V)$ *and* $\alpha \notin \mathsf{Suc}$ *and arbitrary proposition* A *we have*

$$\mathcal{K}, \alpha \Vdash A^{\Box\downarrow} \iff \tilde{\mathcal{K}}, \alpha \Vdash A^{\Box\downarrow}.$$

Proof. First observe that for every $\alpha \in \mathsf{Suc}$ and every proposition B we have

$$\tilde{\mathcal{K}}, \alpha \Vdash B \iff \tilde{\mathcal{K}}, \alpha \models_c B \iff \mathcal{K}, \alpha \models_c B$$

Then we may use Corollary 3.41 and for $\alpha \in \mathsf{Suc}$ deduce

$$\mathcal{K}, \alpha \Vdash B^{\Box} \iff \tilde{\mathcal{K}}, \alpha \Vdash B^{\Box}. \tag{8.2}$$

We use induction on the complexity of A and prove the assertion of the lemma. All cases are obvious except for the cases $A = \Box B$ in which we have $A^{\Box\downarrow} = \Box B^{\Box}$. We have

$$\begin{aligned} \mathcal{K}, \alpha \nVdash \Box B^{\Box} &\iff \text{there exists some } \beta \sqsupset \alpha \text{ such that } \mathcal{K}, \beta \nVdash B^{\Box} \\ &\iff \text{there exists some } \beta \sqsupset \alpha \text{ such that } \tilde{\mathcal{K}}, \beta \nVdash B^{\Box} \\ &\iff \tilde{\mathcal{K}}, \alpha \nVdash \Box B^{\Box} \end{aligned}$$

in which in the second line we use eq. (8.2). ❑

Lemma 8.13. *For every* $A \in \mathcal{L}_{\Box}$ *we have* $\mathsf{iGL\overline{C}T\underline{P}C_a} \vdash A$ *iff* $\mathsf{GLC_a} \vdash A^{\Box\downarrow}$.

Proof. We use induction on the proof $\mathsf{iGL\overline{C}T\underline{P}C_a} \vdash A$ and show $\mathsf{GLC_a} \vdash A^{\Box\downarrow}$. All cases are similar to the one for $\mathsf{iGL\overline{C}T\underline{P}}$, except for

- $A = p \to \Box p$: then $\mathsf{iK4} \vdash A^{\Box\downarrow} \leftrightarrow A$ and hence $\mathsf{GLC_a} \vdash A^{\Box\downarrow}$.

For the other way around, let $\mathsf{iGL\overline{C}T\underline{P}C_a} \nvdash A$. Then by $\Box\mathsf{CP}$ we have $A^{\Box\downarrow} \leftrightarrow A$, and then we may deduce $\mathsf{iGL\overline{C}T\underline{P}C_a} \nvdash A^{\Box\downarrow}$. By Theorem 8.6, there exists some quasi-classical perfect Kripke model $\mathcal{K}$ such that $\mathcal{K}, \alpha \not\models A^{\Box\downarrow}$. Corollary 3.41 implies $\mathcal{K}, \alpha \not\models_c A^{\Box\downarrow}$, which by soundness of $\mathsf{GLC_a}$ for classical Kripke models with the property of truth-ascending (i.e. if p is true at some node, then it is true also at all accessible nodes), implies $\mathsf{GLC_a} \nvdash A^{\Box\downarrow}$. ❑

Theorem 8.14. $\mathsf{iGL\overline{C}T\underline{P}C_a} = \mathcal{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathsf{PA}) \leq \mathcal{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{PA}) = \mathsf{GLC_a}$.

Proof. The arithmetical soundness of $\mathsf{iGL\overline{C}T\underline{P}C_a}$ is straightforward and left to the reader. Also $\mathsf{GLC_a} = \mathcal{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{PA})$ holds by Theorem 7.1. It is enough here to show that

$$\mathcal{AC}_{\Sigma_1}(\mathsf{iGL\overline{C}TC_a}; \mathsf{PA}^*, \mathsf{PA}) \leq_{f,\bar{f}} \mathcal{AC}_{\Sigma_1}(\mathsf{GLC_a}; \mathsf{PA}, \mathsf{PA}).$$

Given $A \in \mathcal{L}_\Box$, let $f(A) := (A)^{\Box\downarrow}$ and $\bar{f}_A$ as identity function.

R1. Lemma 8.13.

R2. If $\mathsf{PA} \nvdash \sigma_{\mathsf{PA}}(A^{\Box\downarrow})$, for a Σ_1-substitution σ, then by Lemma 3.20 we have $\mathsf{PA} \nvdash \sigma_{\mathsf{PA}^*}(A)$. ❑

Lemma 8.15. *For every $A \in \mathcal{L}_\Box$ we have $\mathsf{iGL\overline{CT}C_a} \vdash A$ iff $\mathsf{iGL\overline{P}C_a} \vdash A^{\Box\downarrow}$.*

Proof. We use induction on the proof $\mathsf{iGL\overline{CT}C_a} \vdash A$ and show $\mathsf{iGL\overline{P}C_a} \vdash A^{\Box\downarrow}$. All cases are identical to the corresponding on in the previous proof, except for when $A = p \to \Box p$, which trivially we have $\mathsf{iGL\overline{P}C_a} \vdash A^{\Box\downarrow}$.

For the other way around, let $\mathsf{iGL\overline{CT}C_a} \nvdash A$. Then by Lemma 3.6 we have $A^{\Box\downarrow} \leftrightarrow A$, and hence $\mathsf{iGL\overline{CT}C_a} \nvdash A^{\Box\downarrow}$. By Theorem 8.4, there exists some Suc-quasi-classical semi-perfect atom-complete Kripke model $\mathcal{K}$ such that $\mathcal{K}, \alpha \nVdash A^{\Box\downarrow}$, for some node α. We may assume $\alpha \notin \mathsf{Suc}$, otherwise eliminate all nodes not in $(\alpha \preccurlyeq) \cup (\alpha \sqsubset)$ and consider this new Kripke model instead of $\mathcal{K}$. Obviously the new Kripke model still refutes $A^{\Box\downarrow}$ at α and is Suc-quasi-classical semi-perfect and atom-complete. Hence Lemma 8.12 implies that $\tilde{\mathcal{K}}, \alpha \nVdash A^{\Box\downarrow}$, in which $\tilde{\mathcal{K}}$ indicates the Kripke model derived from $\mathcal{K}$ by making every $\sqsubset$-accessible node as a classical node. Precise definition of $\tilde{\mathcal{K}}$ came before Lemma 8.12. It is obvious that $\tilde{\mathcal{K}}$ is a Suc-classical semi-perfect atom-complete Kripke model. Hence Theorem 7.13 implies $\mathsf{iGL\overline{P}C_a} \nvdash A^{\Box\downarrow}$, as desired. ❑

Theorem 8.16. $\mathsf{iGL\overline{CT}C_a} = \mathcal{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathsf{HA}) \leq \mathcal{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{HA}) = \mathsf{iGL\overline{P}C_a}$.

Proof. The arithmetical soundness of $\mathsf{iGL\overline{CT}C_a}$ is straightforward and left to the reader. Also $\mathsf{iGL\overline{P}C_a} = \mathcal{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{HA})$ holds by Theorem 7.14. It is enough here to show that

$$\mathcal{AC}_{\Sigma_1}(\mathsf{iGL\overline{CT}C_a}; \mathsf{PA}^*, \mathsf{HA}) \leq_{f,\bar{f}} \mathcal{AC}_{\Sigma_1}(\mathsf{iGL\overline{P}C_a}; \mathsf{PA}, \mathsf{HA}).$$

Given $A \in \mathcal{L}_\Box$, let $f(A) := (A)^{\Box\downarrow}$ and $\bar{f}_A$ as identity function.

R1. Lemma 8.15.

R2. If $\mathsf{HA} \nvdash \sigma_{\mathsf{PA}}(A^{\Box\downarrow})$, for a Σ_1-substitution σ, then by Lemma 3.20 we have $\mathsf{HA} \nvdash \sigma_{\mathsf{PA}^*}(A)$. ❑

Lemma 8.17. *For every $A \in \mathcal{L}_\Box$ we have $\mathsf{iGL\overline{C}T\underline{S^*P}C_a} \vdash A$ iff $\mathsf{GL\underline{S}C_a} \vdash A^{\Box\downarrow}$.*

Proof. We use induction on the proof $\mathsf{iGL\overline{C}T\underline{S^*P}C_a} \vdash A$ and show $\mathsf{GL\underline{S}C_a} \vdash A^{\Box\downarrow}$. All cases are similar to the one for $\mathsf{iGL\overline{C}T\underline{S^*P}}$, except for

- $A = p \to \Box p$: then $\mathsf{iK4} \vdash A^{\Box\downarrow} \leftrightarrow A$ and hence $\mathsf{GLC_a} \vdash A^{\Box\downarrow}$.

For the other way around, let $\mathsf{iGL\overline{C}T\underline{S}^*\underline{P}C_a} \nvdash A$. Then by $\Box\mathsf{CP}$ we have $A^{\Box\downarrow} \leftrightarrow A$, and then we may deduce $\mathsf{iGL\overline{C}T\underline{S}^*\underline{P}C_a} \nvdash A^{\Box\downarrow}$. By Theorem 8.8, there exists some quasi-classical perfect Kripke model $\mathcal{K}$ such that $\mathcal{K}, \alpha \not\models A^{\Box\downarrow}$ and $\mathcal{K}$ is $A^{\Box\downarrow}$-sound at α. Corollary 3.41 implies $\mathcal{K}, \alpha \not\models_c A^{\Box\downarrow}$, which by soundness of $\mathsf{GL\underline{S}C_a}$ for classical Kripke models with the property of truth-ascending (i.e. if p is true at some node, then it is true also at all accessible nodes), implies $\mathsf{GL\underline{S}C_a} \nvdash A^{\Box\downarrow}$. ❑

Theorem 8.18. $\mathsf{iGL\overline{C}T\underline{S}^*\underline{P}C_a} = \mathcal{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathbb{N}) \leq \mathcal{PL}_{\Sigma_1}(\mathsf{PA}, \mathbb{N}) = \mathsf{GL\underline{S}C_a}$.

Proof. The arithmetical soundness of $\mathsf{iGL\overline{C}T\underline{S}^*\underline{P}C_a}$ is straightforward and left to the reader. Also $\mathcal{PL}_{\Sigma_1}(\mathsf{PA}, \mathbb{N}) = \mathsf{GL\underline{S}C_a}$ holds by Theorem 7.1. It is enough here to show that

$$\mathcal{AC}_{\Sigma_1}(\mathsf{iGL\overline{C}T\underline{S}^*\underline{P}C_a}; \mathsf{PA}^*, \mathbb{N}) \leq_{f,\bar{f}} \mathcal{AC}_{\Sigma_1}(\mathsf{GL\underline{S}C_a}; \mathsf{PA}, \mathbb{N}).$$

Given $A \in \mathcal{L}_\Box$, let $f(A) := (A)^{\Box\downarrow}$ and $\bar{f}_A$ as identity function.

R1. Lemma 8.17.

R2. If $\mathbb{N} \not\models \sigma_{\mathsf{PA}}(A^{\Box\downarrow})$, for a Σ_1-substitution σ, then by Lemma 3.20 we have $\mathbb{N} \not\models \sigma_{\mathsf{PA}^*}(A)$. ❑

Theorem 8.19. $\mathsf{iGL\overline{C}T\underline{P}C_a} = \mathcal{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathsf{PA}) \leq \mathcal{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathsf{HA}) = \mathsf{iGL\overline{C}\overline{T}C_a}$.

Proof. We already have $\mathcal{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathsf{PA}) = \mathsf{iGL\overline{C}T\underline{P}C_a}$ and $\mathsf{iGL\overline{C}T\underline{P}C_a} = \mathcal{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathsf{PA})$ by Theorems 8.14 and 8.16. It is enough here to show that $\mathcal{AC}_{\Sigma_1}(\mathsf{iGL\overline{C}T\underline{P}C_a}; \mathsf{PA}^*, \mathsf{PA}) \leq_{f,\bar{f}} \mathcal{AC}_{\Sigma_1}(\mathsf{iGL\overline{C}\overline{T}C_a}; \mathsf{PA}^*, \mathsf{HA})$. Given $A \in \mathcal{L}_\Box$, let $f(A) := (A)^{\neg\uparrow}$ and $\bar{f}_A$ as identity function.

R1. If $\mathsf{iGL\overline{C}\overline{T}C_a} \vdash A^{\neg\uparrow}$ then $\mathsf{iGL\overline{C}T\underline{P}C_a} \vdash A^{\neg\uparrow}$, and since we have PEM in $\mathsf{iGL\overline{C}T\underline{P}C_a}$, we may conclude $\mathsf{iGL\overline{C}T\underline{P}C_a} \vdash A$.

R2. If $\mathsf{HA} \nvdash \sigma_{\mathsf{PA}^*}(A^{\neg\uparrow})$, for a Σ_1-substitution σ, then by Lemma 5.18 we have $\mathsf{HA} \nvdash (\sigma_{\mathsf{PA}^*}(A))^{\neg}$. Hence by Lemma 5.15 we have $\mathsf{PA} \nvdash \sigma_{\mathsf{PA}^*}(A)$. ❑

Lemma 8.20. *For every $A \in \mathcal{L}_\Box$, if $\mathsf{iGL\overline{C}T\underline{P}C_a} \vdash A^{\Box\uparrow}$, then $\mathsf{iGLCT} \vdash A$.*

Proof. Let $\mathsf{iGLCT} \nvdash A$. Hence by Theorem 3.38, there is some perfect quasi-classical Kripke model $\mathcal{K}$ such that $\mathcal{K}, \alpha \not\Vdash A$. Then Corollary 3.42 implies $\mathcal{K}, \alpha \not\models A^{\Box\uparrow}$, and hence by soundness of $\mathsf{iGL\overline{C}T\underline{P}C_a}$ (Theorem 8.6) implies $\mathsf{iGL\overline{C}T\underline{P}C_a} \nvdash A^{\Box\uparrow}$. ❑

Theorem 8.21. $\mathsf{iGLCT} = \mathcal{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathsf{PA}^*) \leq \mathcal{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathsf{PA}) = \mathsf{iGL\overline{C}T\underline{P}C_a}$.

Proof. The soundness of iGLCT is straightforward and left to the reader. By Theorem 8.14 we have $\mathcal{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathsf{PA}) = \mathsf{iGL\overline{C}T\underline{P}C_a}$. We must show $\mathcal{AC}_{\Sigma_1}(\mathsf{iGLCT}; \mathsf{PA}^*, \mathsf{PA}^*) \leq_{f,\bar{f}} \mathcal{AC}_{\Sigma_1}(\mathsf{iGL\overline{C}T\underline{P}C_a}; \mathsf{PA}^*, \mathsf{PA})$. Given $A \in \mathcal{L}_\Box$, define $f(A) = A^{\Box\uparrow}$ and $\bar{f}_A$ as identity function.

R1. Lemma 8.20.

R2. Let $\mathsf{PA} \nvdash \sigma_{\mathsf{PA}^*}(A^{\Box\uparrow})$. Then by Lemma 3.17, $\mathsf{PA} \nvdash \sigma_{\mathsf{PA}^*}(A)^{\mathsf{PA}}$, and hence by definition of PA^*, we have $\mathsf{PA}^* \nvdash \sigma_{\mathsf{PA}^*}(A)$. ❑

Corollary 8.22. *For every $A \in \mathcal{L}_\Box$, we have $\mathsf{iGLCT} \vdash A$ iff $\mathsf{iGL\overline{C}T\underline{P}C_a} \vdash A^{\Box\uparrow}$.*

Proof. Use Corollary 4.5 and theorem 8.21. ❑

Theorem 8.23. $\mathsf{iGLCT} = \mathcal{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathsf{PA}^*) \leq \mathcal{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathbb{N}) = \mathsf{iGL\overline{C}T\underline{S}^*\underline{P}C_a}$.

Proof. By Theorems 8.18 and 8.21 we have $\mathcal{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathbb{N}) = \mathsf{iGL\overline{C}T\underline{S}^*\underline{P}C_a}$ and $\mathcal{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathsf{PA}^*) = \mathsf{iGLCT}$. We must show $\mathcal{AC}_{\Sigma_1}(\mathsf{iGLCT}; \mathsf{PA}^*, \mathsf{PA}^*) \leq_{f,\bar{f}} \mathcal{AC}_{\Sigma_1}(\mathsf{iGL\overline{C}T\underline{S}^*\underline{P}C_a}; \mathsf{PA}^*, \mathbb{N})$. Given $A \in \mathcal{L}_\Box$, define $f(A) = \Box A$ and $\bar{f}_A$ as identity function.

R1. Let $\mathsf{iGL\overline{C}T\underline{S^*P}C_a} \vdash \Box A$. By soundness of $\mathsf{iGL\overline{C}T\underline{S^*P}C_a} = \mathcal{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathbb{N})$, for every Σ_1-substitution σ we have $\mathbb{N} \models \sigma_{\mathsf{PA}^*}(\Box A)$ and hence $\mathsf{PA}^* \vdash \sigma_{\mathsf{PA}^*}(A)$. Then by arithmetical completeness of $\mathsf{iGLCT} = \mathcal{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathsf{PA}^*)$, we have $\mathsf{iGLCT} \vdash A$.
One also may prove this item with a direct propositional argument, using Kripke semantics. For simplicity reasons, we chose the indirect way.

R2. Let $\mathbb{N} \not\models \sigma_{\mathsf{PA}^*}(\Box A)$. Then $\mathsf{PA}^* \nvdash \sigma_{\mathsf{PA}^*}(A)$, as desired. ❑

Lemma 8.24. *For every $A \in \mathcal{L}_\Box$, we have $\mathsf{iGL\overline{CT}} \vdash A$ iff $\mathsf{iGL\overline{P}} \vdash A^{\Box\downarrow}$.*

Proof. We use induction on the proof $\mathsf{iGL\overline{CT}} \vdash A$ and show $\mathsf{iGL\overline{P}} \vdash A^{\Box\downarrow}$:

- $\mathsf{iGL} \vdash A$: by Lemma 3.7 we have $\mathsf{iGL} \vdash A^{\Box\downarrow}$.
- A is an axiom instance of $\Box\mathsf{CP}$ or $\Box\mathsf{TP}$: Then $A = \Box B$ and $\mathsf{iGLCT} \vdash B$ and by Lemma 8.9 we have $\mathsf{GL} \vdash B^{\Box}$. By Lemma 8.10 we have $\mathsf{iGL\overline{P}} \vdash \Box B^{\Box}$.
- $\mathsf{iGL\overline{CT}} \vdash B$ and $\mathsf{iGL\overline{CT}} \vdash B \to A$ with lower proof length: by induction hypothesis we have $\mathsf{iGL\overline{P}} \vdash B^{\Box\downarrow}$ and $\mathsf{iGL\overline{P}} \vdash B^{\Box\downarrow} \to A^{\Box\downarrow}$, which implies $\mathsf{iGL\overline{P}} \vdash A^{\Box\downarrow}$, as desired.

For the other way around, let $\mathsf{iGL\overline{CT}} \nvdash A$. Then by Lemma 3.6 we have $A^{\Box\downarrow} \leftrightarrow A$, and hence $\mathsf{iGL\overline{CT}} \nvdash A^{\Box\downarrow}$. By Theorem 8.3, there exists some Suc-quasi-classical semi-perfect Kripke model $\mathcal{K}$ such that $\mathcal{K}, \alpha \nVdash A^{\Box\downarrow}$, for some node α. We may let $\alpha \notin \mathsf{Suc}$, otherwise eliminate all nodes not in $(\alpha \preccurlyeq) \cup (\alpha \sqsubset)$ and consider this new Kripke model instead of $\mathcal{K}$. Obviously the new Kripke model still refutes $A^{\Box\downarrow}$ at α and is Suc-quasi-classical semi-perfect. Hence Lemma 8.12 implies that $\tilde{\mathcal{K}}, \alpha \nVdash A^{\Box\downarrow}$, in which $\tilde{\mathcal{K}}$ indicates the Kripke model derived from $\mathcal{K}$ by making every $\sqsubset$-accessible node as a classical node. Precise definition of $\tilde{\mathcal{K}}$ came before Lemma 8.12. It is obvious that $\tilde{\mathcal{K}}$ is a Suc-classical semi-perfect Kripke model. Hence Theorem 7.12 implies $\mathsf{iGL\overline{P}} \nvdash A^{\Box\downarrow}$, as desired. ❑

Theorem 8.25. $\mathsf{iGL\overline{CT}} = \mathcal{PL}(\mathsf{PA}^*, \mathsf{HA}) \leq \mathcal{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathsf{HA}) = \mathsf{iGL\overline{CT}C_a}$.

Proof. The arithmetical soundness of $\mathsf{iGL\overline{C}TC_a}$ is straightforward and left to the reader. By Theorem 8.16 we have $\mathcal{PL}(\mathsf{PA}^*, \mathsf{HA}) = \mathsf{iGL\overline{CT}C_a}$. We must show

$$\mathcal{AC}(\mathsf{iGL\overline{CT}}; \mathsf{PA}^*, \mathsf{HA}) \leq_{f,\bar{f}} \mathcal{AC}_{\Sigma_1}(\mathsf{iGL\overline{CT}C_a}; \mathsf{PA}^*, \mathsf{HA}).$$

Given $A \in \mathcal{L}_\Box$, if $\mathsf{iGL\overline{CT}} \vdash A$, define $f(A) := \top$. If $\mathsf{iGL\overline{CT}} \nvdash A$, by Lemma 8.24 we have $\mathsf{iGL\overline{P}} \nvdash A^{\Box\downarrow}$, and hence by Lemma 7.23 there exists some propositional $(.)^{\Box\downarrow}$-substitution τ such that $\mathsf{iGL\overline{P}C_a} \nvdash \tau(A^{\Box\downarrow})$. Define $f(A) := \tau(A)$ and $\bar{f}_A(\sigma) := \sigma_{\mathsf{PA}^*} \circ \tau$.

R1. Let $\mathsf{iGL\overline{CT}} \nvdash A$. By Lemma 8.24 we have $\mathsf{iGL\overline{P}} \nvdash A^{\Box\downarrow}$ and then Lemma 7.23 implies $\mathsf{iGL\overline{P}C_a} \nvdash \tau(A^{\Box\downarrow})$, in which τ is as used for the definition of $f(A)$. Since τ is a $(.)^{\Box\downarrow}$-substitution, by Lemma 7.3 we have $\mathsf{iGL\overline{P}C_a} \nvdash (\tau(A))^{\Box\downarrow}$. Then Lemma 8.15 implies that $\mathsf{iGL\overline{CT}C_a} \nvdash \tau(A)$, or in other words $\mathsf{iGL\overline{CT}C_a} \nvdash f(A)$.

R2. Let $\mathsf{HA} \nvdash \sigma_{\mathsf{PA}^*}(f(A))$ for some Σ_1-substitution σ. By definition of $f(A)$, we must have $\mathsf{iGL\overline{CT}} \nvdash A$, otherwise $f(A) := \top$, which contradicts $\mathsf{HA} \nvdash \sigma_{\mathsf{PA}^*}(f(A))$. Hence $f(A) = \tau(A)$ for some propositional $(.)^{\Box\downarrow}$-substitution τ. By Lemma 3.19 we have $\mathsf{HA} \nvdash \sigma_{\mathsf{PA}}(\tau(A)^{\Box\downarrow})$. Since $\mathsf{iK4} + \mathsf{CP_a}$ is included in $\mathcal{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{HA}) = \mathsf{iGL\overline{P}C_a}$ (Theorem 7.21), we have $\mathsf{HA} \nvdash \sigma_{\mathsf{PA}}(\tau(A^{\Box\downarrow}))$. This implies that $\mathsf{HA} \nvdash [\bar{f}_A(\sigma)]_{\mathsf{PA}}(A^{\Box\downarrow})$ and again by Lemma 3.19 we have $\mathsf{HA} \nvdash [\bar{f}_A(\sigma)]_{\mathsf{PA}^*}(A)$. ❑

Lemma 8.26. *For every $A \in \mathcal{L}_\Box$, we have $\mathsf{iGL\overline{C}T\underline{P}} \vdash A$ iff $\mathsf{GL} \vdash A^{\Box\downarrow}$.*

Proof. We use induction on the proof $\mathsf{iGL\overline{C}T\underline{P}} \vdash A$ and show $\mathsf{GL} \vdash A^{\Box\downarrow}$:

- $A = \Box B$ and $\mathsf{iGLCT} \vdash B$: by Lemma 8.9 we have $\mathsf{GL} \vdash B^{\Box}$ and hence by necessitation $\mathsf{GL} \vdash \Box B^{\Box}$.

- $\mathsf{iGL} \vdash A$: by Lemma 3.7 we have $\mathsf{iGL} \vdash A^{\Box\downarrow}$.
- $A = B \vee \neg B$: Then $A^{\Box\downarrow} = B^{\Box\downarrow} \vee \neg B^{\Box\downarrow}$ which is valid in GL.
- $\mathsf{iGL\overline{C}T\underline{P}} \vdash B$ and $\mathsf{iGL\overline{C}T\underline{P}} \vdash B \to A$ with lower proof length than the one for A: by induction hypothesis we have $\mathsf{GL} \vdash B^{\Box\downarrow}$ and $\mathsf{GL} \vdash B^{\Box\downarrow} \to A^{\Box\downarrow}$, which implies $\mathsf{GL} \vdash A^{\Box\downarrow}$, as desired.

For the other way around, let $\mathsf{iGL\overline{C}T\underline{P}} \nvdash A$. Then by $\Box\mathsf{CP}$ we have $A^{\Box\downarrow} \leftrightarrow A$, and then we may deduce $\mathsf{iGL\overline{C}T\underline{P}} \nvdash A^{\Box\downarrow}$. By Theorem 8.5, there exists some quasi-classical perfect Kripke model $\mathcal{K}$ and some boolean interpretation I such that $\mathcal{K}, \alpha, I \not\models A^{\Box\downarrow}$. Corollary 3.41 implies $\mathcal{K}, \alpha, I \not\models_c A^{\Box\downarrow}$, which by soundness of GL for classical Kripke models, implies $\mathsf{GL} \nvdash A^{\Box\downarrow}$. ❑

Theorem 8.27. $\mathsf{iGL\overline{C}T\underline{P}} = \mathcal{PL}(\mathsf{PA}^*, \mathsf{PA}) \leq \mathcal{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathsf{PA}) = \mathsf{iGL\overline{C}T\underline{P}C_a}$.

Proof. The arithmetical soundness of $\mathsf{iGL\overline{C}T\underline{P}}$ is straightforward and left to the reader. By Theorem 8.14 we have $\mathcal{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathsf{PA}) = \mathsf{iGL\overline{C}T\underline{P}C_a}$. We must show

$$\mathcal{AC}(\mathsf{iGL\overline{C}T\underline{P}}; \mathsf{PA}^*, \mathsf{PA}) \leq_{f,\bar{f}} \mathcal{AC}_{\Sigma_1}(\mathsf{iGL\overline{C}T\underline{P}C_a}; \mathsf{PA}^*, \mathsf{PA}).$$

Given $A \in \mathcal{L}_\Box$, if $\mathsf{iGL\overline{C}T\underline{P}} \vdash A$, define $f(A) := \top$. If $\mathsf{iGL\overline{C}T\underline{P}} \nvdash A$, by Lemma 8.26 we have $\mathsf{GL} \nvdash A^{\Box\downarrow}$, and hence by Remark 7.4 there exists some propositional $(.)^{\Box\downarrow}$-substitution τ such that $\mathsf{GLC_a} \nvdash \tau(A^{\Box\downarrow})$. Define $f(A) := \tau(A)$ and $\bar{f}_A(\sigma) := \sigma_{\mathsf{PA}^*} \circ \tau$.

R1. Let $\mathsf{iGL\overline{C}T\underline{P}} \nvdash A$. By Lemma 8.26 we have $\mathsf{GL} \nvdash A^{\Box\downarrow}$ and then Remark 7.4 implies $\mathsf{GLC_a} \nvdash \tau(A^{\Box\downarrow})$, in which τ is as used for the definition of $f(A)$. Since τ is a $(.)^{\Box\downarrow}$-substitution, by Lemma 7.3 we have $\mathsf{GLC_a} \nvdash (\tau(A))^{\Box\downarrow}$. Then Lemma 8.13 implies that $\mathsf{iGL\overline{C}T\underline{P}C_a} \nvdash \tau(A)$, or in other words $\mathsf{iGL\overline{C}T\underline{P}C_a} \nvdash f(A)$.

R2. Let $\mathsf{PA} \nvdash \sigma_{\mathsf{PA}^*}(f(A))$ for some Σ_1-substitution σ. By definition of $f(A)$, we must have $\mathsf{iGL\overline{C}T\underline{P}} \nvdash A$, otherwise $f(A) := \top$, which contradicts $\mathsf{PA} \nvdash \sigma_{\mathsf{PA}^*}(f(A))$. Hence $f(A) = \tau(A)$ for some propositional $(.)^{\Box\downarrow}$-substitution τ. By Lemma 3.19 we have $\mathsf{PA} \nvdash \sigma_{\mathsf{PA}}(\tau(A)^{\Box\downarrow})$. Since $\mathsf{iK4} + \mathsf{CP_a}$ is included in $\mathcal{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{PA}) = \mathsf{GLC_a}$ (Theorem 7.1), by Lemma 7.3 we have $\mathsf{PA} \nvdash \sigma_{\mathsf{PA}}(\tau(A^{\Box\downarrow}))$. This implies that $\mathsf{PA} \nvdash [\bar{f}_A(\sigma)]_{\mathsf{PA}}(A^{\Box\downarrow})$ and again by Lemma 3.19 we have $\mathsf{PA} \nvdash [\bar{f}_A(\sigma)]_{\mathsf{PA}^*}(A)$. ❑

Lemma 8.28. *For every $A \in \mathcal{L}_\Box$, we have* $\mathsf{iGLCT} \vdash A$ *iff* $\mathsf{GL} \vdash A^\Box$.

Proof. One may use induction on the proof $\mathsf{iGLCT} \vdash A$ to show that $\mathsf{GL} \vdash A^\Box$. For the other direction, we reason contrapositively. Let $\mathsf{iGLCT} \nvdash A$. Since in iGLC we have $A \leftrightarrow A^\Box$, we have $\mathsf{iGLCT} \nvdash A^\Box$. Hence by Theorem 3.38 there is some perfect quasi-classical model $\mathcal{K}$ such that $\mathcal{K}, \alpha \nVdash A^\Box$. Hence by Corollary 3.41 $\mathcal{K}, \alpha \not\models_c A^\Box$. Since $\models_c$ is just a classical semantics for the modal logic GL, by the soundness of GL for finite irreflexive Kripke models [Smo85, Chapter 2.2], we may deduce $\mathsf{GL} \nvdash A^\Box$, as desired. ❑

Lemma 8.29. *For every $A \in \mathcal{L}_\Box$, we have* $\mathsf{iGLCT} \vdash A$ *iff* $\mathsf{GLC_a} \vdash A^\Box$.

Proof. There are two options (atleast) for the proof. First is that one repeat a similar argument of the proof o for Lemma 8.28. Second proof follows: By Corollary 8.22, $\mathsf{iGLCT} \vdash A$ iff $\mathsf{iGL\overline{C}T\underline{P}C_a} \vdash A^{\Box\uparrow}$, and Lemma 8.13 implies $\mathsf{iGL\overline{C}T\underline{P}C_a} \vdash A^{\Box\uparrow}$ iff $\mathsf{GLC_a} \vdash (A^{\Box\uparrow})^{\Box\downarrow}$. Since $\mathsf{iK4} \vdash A^\Box \leftrightarrow (A^{\Box\uparrow})^{\Box\downarrow}$, we have the desired result. ❑

Theorem 8.30. $\mathsf{iGLCT} = \mathcal{PL}(\mathsf{PA}^*, \mathsf{PA}^*) \leq \mathcal{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathsf{PA}^*) = \mathsf{iGLCT}$.

Proof. The arithmetical soundness of iGLCT for general substitutions, i.e. $\mathcal{AS}(\mathsf{iGLCT};\mathsf{PA}^*,\mathsf{PA}^*)$, is straightforward and left to the reader. By Theorem 8.21 we have $\mathcal{PL}_{\Sigma_1}(\mathsf{PA}^*,\mathsf{PA}^*)=\mathsf{iGLCT}$. We must show

$$\mathcal{AC}(\mathsf{iGLCT};\mathsf{PA}^*,\mathsf{PA}^*)\leq_{f,\bar{f}}\mathcal{AC}_{\Sigma_1}(\mathsf{iGLCT};\mathsf{PA}^*,\mathsf{PA}^*).$$

Given $A\in\mathcal{L}_\Box$, if $\mathsf{iGLCT}\vdash A$, define $f(A):=\top$. If $\mathsf{iGLCT}\nvdash A$, by Lemma 8.28 we have $\mathsf{GL}\nvdash A^\Box$, and hence by Remark 7.4 there exists some propositional $(.)^{\Box\!\downarrow}$-substitution τ such that $\mathsf{GLC_a}\nvdash\tau(A^\Box)$. Define $f(A):=\tau(A)$ and $\bar{f}_A(\sigma):=\sigma_{\mathsf{PA}^*}\circ\tau$.

R1. Let $\mathsf{iGLCT}\nvdash A$. By Lemma 8.28 we have $\mathsf{GL}\nvdash A^\Box$ and then Remark 7.4 implies $\mathsf{GLC_a}\nvdash\tau(A^\Box)$, in which τ is as used for the definition of $f(A)$. Since τ is a $(.)^{\Box\!\downarrow}$-substitution, by Lemma 7.3 we have $\mathsf{GLC_a}\nvdash\tau(A)^\Box$. Then Lemma 8.29 implies that $\mathsf{iGLCT}\nvdash\tau(A)$, or in other words $\mathsf{iGLCT}\nvdash f(A)$.

R2. Let $\mathsf{PA}^*\nvdash\sigma_{\mathsf{PA}^*}(f(A))$ for some Σ_1-substitution σ. By definition of $f(A)$, we must have $\mathsf{iGLCT}\nvdash A$, otherwise $f(A):=\top$, which contradicts $\mathsf{PA}^*\nvdash\sigma_{\mathsf{PA}^*}(f(A))$. Hence $f(A)=\tau(A)$ for some propositional $(.)^{\Box\!\downarrow}$-substitution τ. Then we have $\mathsf{PA}\nvdash\sigma_{\mathsf{PA}^*}(\tau(A))^{\mathsf{PA}}$ and by Lemma 3.18 we have $\mathsf{PA}\nvdash\sigma_{\mathsf{PA}}(\tau(A)^\Box)$. Since $\mathsf{iK4}+\mathsf{CP_a}$ is included in $\mathcal{PL}_{\Sigma_1}(\mathsf{PA},\mathsf{PA})=\mathsf{GLC_a}$ (Theorem 7.1), by Lemma 7.3 we have $\mathsf{PA}\nvdash\sigma_{\mathsf{PA}}(\tau(A^\Box))$. This implies that $\mathsf{PA}\nvdash[\bar{f}_A(\sigma)]_{\mathsf{PA}}(A^\Box)$ and again by Lemma 3.18 we have $\mathsf{PA}\nvdash([\bar{f}_A(\sigma)]_{\mathsf{PA}^*}(A))^{\mathsf{PA}}$. Hence $\mathsf{PA}^*\nvdash[\bar{f}_A(\sigma)]_{\mathsf{PA}^*}(A)$. ❑

Lemma 8.31. *For every $A\in\mathcal{L}_\Box$, we have $\mathsf{iGL\overline{C}T\underline{S}^*\underline{P}}\vdash A$ iff $\mathsf{GL\underline{S}}\vdash A^{\Box\!\downarrow}$.*

Proof. We use induction on the proof $\mathsf{iGL\overline{C}T\underline{S}^*\underline{P}}\vdash A$ and show $\mathsf{GL\underline{S}}\vdash A^{\Box\!\downarrow}$. All cases are similar to the one for item 3 above, except for

- $A=\Box B\to B^\Box$: since $\mathsf{iK4}\vdash A^{\Box\!\downarrow}\leftrightarrow(\Box B^\Box\to B^\Box)$, we may deduce $\mathsf{GL\underline{S}}\vdash A^{\Box\!\downarrow}$.

For the other way around, let $\mathsf{iGL\overline{C}T\underline{S}^*\underline{P}}\nvdash A$. Then by $\Box\mathsf{CP}$ we have $A^{\Box\!\downarrow}\leftrightarrow A$, and then we may deduce $\mathsf{iGL\overline{C}T\underline{S}^*\underline{P}}\nvdash A^{\Box\!\downarrow}$. By Theorem 8.7, there exists some quasi-classical perfect Kripke model $\mathcal{K}$ and some boolean interpretation I such that $\mathcal{K},\alpha,I\nVdash A^{\Box\!\downarrow}$ and $\mathcal{K}$ is $A^{\Box\!\downarrow}$-sound at α. Corollary 3.41 implies $\mathcal{K},\alpha,I\nVdash_c A^{\Box\!\downarrow}$, which by soundness of $\mathsf{GL\underline{S}}$ (restricted to sub-formulas of $A^{\Box\!\downarrow}$) for $A^{\Box\!\downarrow}$-sound classical Kripke models, implies $\mathsf{GL\underline{S}}\nvdash A^{\Box\!\downarrow}$. ❑

Theorem 8.32. $\mathsf{iGL\overline{C}T\underline{S}^*\underline{P}}=\mathcal{PL}(\mathsf{PA}^*,\mathbb{N})\leq\mathcal{PL}_{\Sigma_1}(\mathsf{PA}^*,\mathbb{N})=\mathsf{iGL\overline{C}T\underline{S}^*\underline{P}C_a}$.

Proof. The arithmetical soundness of $\mathsf{iGL\overline{C}T\underline{S}^*\underline{P}}$ is straightforward and left to the reader. By Theorem 8.18 we have $\mathcal{PL}_{\Sigma_1}(\mathsf{PA}^*,\mathbb{N})=\mathsf{iGL\overline{C}T\underline{S}^*\underline{P}C_a}$. We must show

$$\mathcal{AC}(\mathsf{iGL\overline{C}T\underline{S}^*\underline{P}};\mathsf{PA}^*,\mathbb{N})\leq_{f,\bar{f}}\mathcal{AC}_{\Sigma_1}(\mathsf{iGL\overline{C}T\underline{S}^*\underline{P}C_a};\mathsf{PA}^*,\mathbb{N}).$$

Given $A\in\mathcal{L}_\Box$, if $\mathsf{iGL\overline{C}T\underline{S}^*\underline{P}}\vdash A$, define $f(A):=\top$. If $\mathsf{iGL\overline{C}T\underline{S}^*\underline{P}}\nvdash A$, by Lemma 8.31 we have $\mathsf{GL\underline{S}}\nvdash A^{\Box\!\downarrow}$, and hence by Remark 7.4 there exists some propositional $(.)^{\Box\!\downarrow}$-substitution τ such that $\mathsf{GL\underline{S}C_a}\nvdash\tau(A^{\Box\!\downarrow})$. Define $f(A):=\tau(A)$ and $\bar{f}_A(\sigma):=\sigma_{\mathsf{PA}^*}\circ\tau$.

R1. Let $\mathsf{iGL\overline{C}T\underline{S}^*\underline{P}}\nvdash A$. By Lemma 8.31 we have $\mathsf{GL\underline{S}}\nvdash A^{\Box\!\downarrow}$ and then Remark 7.4 implies $\mathsf{GL\underline{S}C_a}\nvdash\tau(A^{\Box\!\downarrow})$, in which τ is as used for the definition of $f(A)$. Since τ is a $(.)^{\Box\!\downarrow}$-substitution, by Lemma 7.3 we have $\mathsf{GL\underline{S}C_a}\nvdash\tau(A)^{\Box\!\downarrow}$. Then Lemma 8.17 implies that $\mathsf{iGL\overline{C}T\underline{S}^*\underline{P}C_a}\nvdash\tau(A)$, or in other words $\mathsf{iGL\overline{C}T\underline{S}^*\underline{P}C_a}\nvdash f(A)$.

R2. Let $\mathbb{N}\not\models\sigma_{\mathsf{PA}^*}(f(A))$ for some Σ_1-substitution σ. By definition of $f(A)$, we must have $\mathsf{iGL\overline{C}T\underline{S}^*\underline{P}}\nvdash A$, otherwise $f(A):=\top$, which contradicts $\mathbb{N}\not\models\sigma_{\mathsf{PA}^*}(f(A))$. Hence $f(A)=\tau(A)$ for some propositional $(.)^{\Box\!\downarrow}$-substitution τ. We have $\mathbb{N}\not\models\sigma_{\mathsf{PA}^*}(\tau(A))$ and by Lemma 3.19we have $\mathbb{N}\not\models\sigma_{\mathsf{PA}}(\tau(A)^{\Box\!\downarrow})$. Since $\mathsf{iK4}+\mathsf{CP_a}$ is included in $\mathcal{PL}_{\Sigma_1}(\mathsf{PA},\mathbb{N})=\mathsf{GL\underline{S}C_a}$ (Theorem 7.1), by Lemma 7.3 we have $\mathbb{N}\not\models\sigma_{\mathsf{PA}}(\tau(A^{\Box\!\downarrow}))$. This implies that $\mathbb{N}\not\models[\bar{f}_A(\sigma)]_{\mathsf{PA}}(A^{\Box\!\downarrow})$ and again by Lemma 3.19 we have $\mathbb{N}$
not $\models[\bar{f}_A(\sigma)]_{\mathsf{PA}^*}(A)$. ❑

Theorem 8.33. $\mathsf{iGLCT} = \mathcal{PL}(\mathsf{PA}^*, \mathsf{PA}^*) \leq \mathcal{PL}(\mathsf{PA}^*, \mathbb{N}) = \mathsf{iGL\overline{C}T\underline{S^*P}}$.

Proof. By Theorems 8.30 and 8.32 we have $\mathcal{PL}(\mathsf{PA}^*, \mathsf{PA}^*) = \mathsf{iGLCT}$ and $\mathcal{PL}(\mathsf{PA}^*, \mathbb{N}) = \mathsf{iGL\overline{C}T\underline{S^*P}}$. We must show $\mathcal{AC}(\mathsf{iGLCT}; \mathsf{PA}^*, \mathsf{PA}^*) \leq_{f,\bar{f}} \mathcal{AC}(\mathsf{iGL\overline{C}T\underline{S^*P}}; \mathsf{PA}^*, \mathbb{N})$. Given $A \in \mathcal{L}_\Box$, define $f(A) = \Box A$ and $\bar{f}_A$ as identity function.

R1. Let $\mathsf{iGL\overline{C}T\underline{S^*P}} \vdash \Box A$. By soundness of $\mathsf{iGL\overline{C}T\underline{S^*P}} = \mathcal{PL}(\mathsf{PA}^*, \mathbb{N})$, for every substitution σ we have $\mathbb{N} \models \sigma_{\mathsf{PA}^*}(\Box A)$ and hence $\mathsf{PA}^* \vdash \sigma_{\mathsf{PA}^*}(A)$. Then by arithmetical completeness of $\mathsf{iGLCT} = \mathcal{PL}(\mathsf{PA}^*, \mathsf{PA}^*)$, we have $\mathsf{iGLCT} \vdash A$.

R2. Let $\mathbb{N} \not\models \sigma_{\mathsf{PA}^*}(\Box A)$. Then $\mathsf{PA}^* \nvdash \sigma_{\mathsf{PA}^*}(A)$, as desired. ❑

Theorem 8.34. $\mathsf{iGL\overline{C}T\underline{P}} = \mathcal{PL}(\mathsf{PA}^*, \mathsf{PA}) \leq \mathcal{PL}(\mathsf{PA}^*, \mathsf{HA}) = \mathsf{iGL\overline{CT}}$.

Proof. We already have $\mathsf{iGL\overline{CT}} = \mathcal{PL}(\mathsf{PA}^*, \mathsf{HA})$ and $\mathcal{PL}(\mathsf{PA}^*, \mathsf{PA}) = \mathsf{iGL\overline{C}T\underline{P}}$ by Theorems 8.25 and 8.27. It is enough here to show that $\mathcal{AC}(\mathsf{iGL\overline{C}T\underline{P}}; \mathsf{PA}^*, \mathsf{PA}) \leq_{f,\bar{f}} \mathcal{AC}(\mathsf{iGL\overline{CT}}; \mathsf{PA}^*, \mathsf{HA})$. Given $A \in \mathcal{L}_\Box$, let $f(A) := (A)^{\neg\uparrow}$ and $\bar{f}_A$ as identity function.

R1. If $\mathsf{iGL\overline{CT}} \vdash A^{\neg\uparrow}$ then $\mathsf{iGL\overline{C}T\underline{P}} \vdash A^{\neg\uparrow}$, and since we have PEM in $\mathsf{iGL\overline{C}T\underline{P}C_a}$, we may conclude $\mathsf{iGL\overline{C}T\underline{P}C_a} \vdash A$.

R2. If $\mathsf{HA} \nvdash \sigma_{\mathsf{PA}^*}(A^{\neg\uparrow})$, for a substitution σ, then by Lemma 5.18 we have $\mathsf{HA} \nvdash (\sigma_{\mathsf{PA}^*}(A))^{\neg}$. Hence by Lemma 5.15 we have $\mathsf{PA} \nvdash \sigma_{\mathsf{PA}^*}(A)$. ❑

9 Conclusion

From Diagram 7, it turns out that the truth Σ_1-provability logic of HA, is the hardest provability logic among the provability logics in Table 6. Closer inspection in the reductions provided in previous sections, reveals that all propositional reductions, i.e. the functions f, are computable. Hence by decidability of $\mathcal{PL}_{\Sigma_1}(\mathsf{HA}, \mathbb{N})$ (Corollary 5.6) and Theorem 4.8, we have the decidability of all provability logics in Table 6:

Corollary 9.1. *All provability logics in the Table 6, are decidable.*

So far, we have seen many reductions of provability logics. The reductions, helped out to prove new arithmetical completeness results, have a more general view of all provability logics and intuitively say which provability logic is *harder*. The reader may wonder what other reductions hold, beyond the transitive closure of the Diagram 7. However it seems more likely that no other reductions holds, at the moment we can not say anything more than that. This question calls for more work.

Conjecture 9.2. We conjecture that the following characterizations and reductions holds:

1. $\mathsf{iH} = \mathcal{PL}(\mathsf{HA}, \mathsf{HA}) \leq \mathcal{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{HA}) = \mathsf{iH}_\sigma$.
2. $\mathsf{iH} = \mathcal{PL}(\mathsf{HA}, \mathsf{HA}) \leq \mathcal{PL}(\mathsf{HA}, \mathbb{N}) = \mathsf{iH\underline{SP}}$.
3. $\mathsf{iH\underline{P}} = \mathcal{PL}(\mathsf{HA}, \mathsf{PA}) \leq \mathcal{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{PA}) = \mathsf{iH}_\sigma\mathsf{\underline{P}}$.
4. $\mathsf{iH\underline{SP}} = \mathcal{PL}(\mathsf{HA}, \mathbb{N}) \leq \mathcal{PL}_{\Sigma_1}(\mathsf{HA}, \mathbb{N}) = \mathsf{iH}_\sigma\mathsf{\underline{SP}}$.
5. $\mathsf{iH}^* = \mathcal{PL}(\mathsf{HA}^*, \mathsf{HA}^*) \leq \mathcal{PL}_{\Sigma_1}(\mathsf{HA}^*, \mathsf{HA}^*) = \mathsf{iH}_\sigma^{**}$.

Moreover, all reductions are computable and hence all provability logics are conjectured to be decidable. In which

- iH is as defined in [Iem01],

- iH* as defined in [AM19],
- iH$\underline{\mathsf{P}}$ is iH plus $\underline{\mathsf{P}}$,
- iH$\underline{\mathsf{SP}}$ is iH plus $\underline{\mathsf{S}}$ and $\underline{\mathsf{P}}$,

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Appendices

Name(s)	**Axiom Scheme**	**Name(s)**	**Axiom Scheme**
$\underline{\mathsf{K}}$	$\Box(A \to B) \to (\Box A \to \Box B)$	$\underline{\mathsf{4}}$	$\Box A \to \Box\Box A$
$\underline{\mathsf{L\ddot{o}b}}$, $\underline{\mathsf{L}}$	$\Box(\Box A \to A) \to \Box A$	$\underline{\mathsf{CP}}$, $\underline{\mathsf{C}}$	$A \to \Box A$
$\underline{\mathsf{S}}$	$\Box A \to A$	$\underline{\mathsf{CP_a}}$, $\underline{\mathsf{C_a}}$	$p \to \Box p$ for atomic variable p
$\underline{\mathsf{S}^*}$	$\Box A \to A^{\Box}$	$\underline{\mathsf{PEM}}$, $\underline{\mathsf{P}}$	$A \vee \neg A$
$\underline{\mathsf{Le}}$	$\Box(A \vee B) \to \Box(\boxdot A \vee \boxdot B)$	$\underline{\mathsf{Le}^+}$	$\Box A \to \Box A^{l}$
$\underline{\mathsf{TP}}$, $\underline{\mathsf{T}}$	$\Box(A \to B) \to (A \vee (A \to B))$	$\underline{\mathsf{i}}$	All theorems of $\mathsf{IPC}_{\Box}$
		$\underline{\mathsf{V}}$	$A \leftrightarrow A^-$
For an axiom scheme $\underline{\mathsf{A}}$, let $\overline{\mathsf{A}}$ indicates $\Box\,\underline{\mathsf{A}}$ and A indicates $\overline{\mathsf{A}} \wedge \underline{\mathsf{A}}$			

Table 5: List of axiom schemas

Theory	Axioms	Provability Logic(s)	Reference
$\mathsf{iK4}$	$\mathsf{i,K,4}$		
iGL	$\mathsf{iK4,L}$		
GL	$\mathsf{iGL,P}$	$\mathcal{PL}(\mathsf{PA},\mathsf{PA})$	[Sol76]
$\mathsf{GLC_a}$	$\mathsf{GL,CP_a}$	$\mathcal{PL}_{\Sigma_1}(\mathsf{PA},\mathsf{PA})$	[Vis82]
$\mathsf{GL\underline{S}}$	$\mathsf{GL,\underline{S}}$	$\mathcal{PL}(\mathsf{PA},\mathbb{N})$	[Sol76]
$\mathsf{GL\underline{S}C_a}$	$\mathsf{GLC_a,\underline{S}}$	$\mathcal{PL}_{\Sigma_1}(\mathsf{PA},\mathbb{N})$	[Vis82]
iGLCT	$\mathsf{iGL,C,T}$	$\mathcal{PL}(\mathsf{PA}^*,\mathsf{PA}^*)$ $\mathcal{PL}_{\Sigma_1}(\mathsf{PA}^*,\mathsf{PA}^*)$	[Vis82]
iH_σ	$\mathsf{iGL,V,\ Le^+}$	$\mathcal{PL}_{\Sigma_1}(\mathsf{HA},\mathsf{HA})$	[AM18, VZ19]
iH_σ^{**}	$\{A : \mathsf{iH}_\sigma \vdash A^\Box\}$	$\mathcal{PL}_{\Sigma_1}(\mathsf{HA}^*,\mathsf{HA}^*)$	[AM19]
$\mathsf{iH}_\sigma\underline{\mathsf{P}}$	$\mathsf{iH}_\sigma,\underline{\mathsf{P}}$	$\mathcal{PL}_{\Sigma_1}(\mathsf{HA},\mathsf{PA})$	Theorem 5.12
$\mathsf{iH}_\sigma\underline{\mathsf{SP}}$	$\mathsf{iH}_\sigma,\underline{\mathsf{S}},\underline{\mathsf{P}}$	$\mathcal{PL}_{\Sigma_1}(\mathsf{HA},\mathbb{N})$	Theorem 5.13
$\mathsf{iH}_\sigma\underline{\mathsf{SP}}^*$	$\{A : \mathsf{iH}_\sigma\underline{\mathsf{SP}} \vdash A^{\Box\downarrow}\}$	$\mathcal{PL}_{\Sigma_1}(\mathsf{HA}^*,\mathbb{N})$	Theorem 6.1
$\mathsf{iH}_\sigma\underline{\mathsf{P}}^*$	$\{A : \mathsf{iH}_\sigma\underline{\mathsf{P}} \vdash A^{\Box\downarrow}\}$	$\mathcal{PL}_{\Sigma_1}(\mathsf{HA}^*,\mathsf{PA})$	Theorem 6.2
iH_σ^*	$\{A : \mathsf{iH}_\sigma \vdash A^{\Box\downarrow}\}$	$\mathcal{PL}_{\Sigma_1}(\mathsf{HA}^*,\mathsf{HA})$	Theorem 6.3
$\mathsf{iGL\overline{P}C_a}$	$\mathsf{iGL,\overline{P},C_a}$	$\mathcal{PL}_{\Sigma_1}(\mathsf{PA},\mathsf{HA})$	Theorem 7.21
$\mathsf{iGL\overline{P}}$	$\mathsf{iGL,\overline{P}}$	$\mathcal{PL}(\mathsf{PA},\mathsf{HA})$	Theorem 7.25
$\mathsf{iGL\overline{C}T\underline{P}C_a}$	$\mathsf{iGL,\overline{C},T,\underline{P},C_a}$	$\mathcal{PL}_{\Sigma_1}(\mathsf{PA}^*,\mathsf{PA})$	Theorem 8.14
$\mathsf{iGL\overline{C}\overline{T}C_a}$	$\mathsf{iGL,\overline{C},\overline{T},C_a}$	$\mathcal{PL}_{\Sigma_1}(\mathsf{PA}^*,\mathsf{HA})$	Theorem 8.16
$\mathsf{iGL\overline{C}T\underline{S^*P}C_a}$	$\mathsf{iGL,\overline{C},T,\underline{S^*},\underline{P},C_a}$	$\mathcal{PL}_{\Sigma_1}(\mathsf{PA}^*,\mathbb{N})$	Theorem 8.18
$\mathsf{iGL\overline{C}T\underline{P}}$	$\mathsf{iGL,\overline{C},T,\underline{P}}$	$\mathcal{PL}(\mathsf{PA}^*,\mathsf{PA})$	Theorem 8.27
$\mathsf{iGL\overline{C}\overline{T}}$	$\mathsf{iGL,\overline{C},\overline{T}}$	$\mathcal{PL}(\mathsf{PA}^*,\mathsf{HA})$	Theorem 8.25
$\mathsf{iGL\overline{C}T\underline{S^*P}}$	$\mathsf{iGL,\overline{C},T,\underline{S^*},\underline{P}}$	$\mathcal{PL}(\mathsf{PA}^*,\mathbb{N})$	Theorem 8.32

Table 6: List of all provability logics

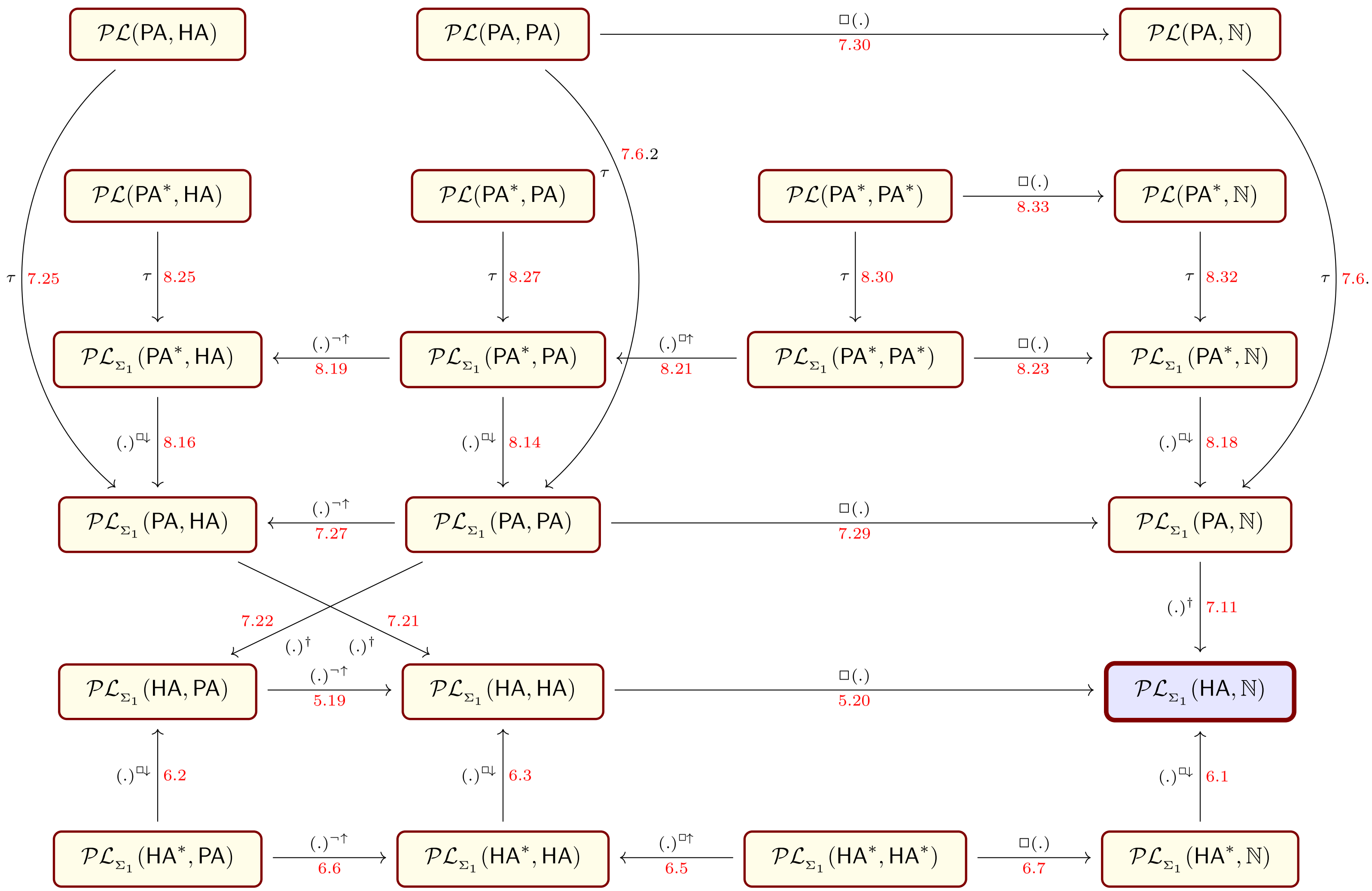

Diagram 7: Reductions of all provability logics. Arrows indicate a reduction of the completeness of the left hand side to the right one. The propositional reduction is shown over the arrow line and the theorem number proving this, is shown under arrow line.