Completeness of intermediate logics with doubly negated axioms

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Received XXXX, revised XXXX, accepted XXXX
Published online XXXX

Key words Intermediate logics, doubly negated axioms.
MSC (2010) 03B55

Let **L** denote a first-order logic in a language that contains infinitely many constant symbols and also containing intuitionistic logic **IQC**. By $\neg\neg$**L**, we mean the associated logic axiomatized by the double negation of the universal closure of the axioms of **L** plus **IQC**. We will show that if **L** is *strongly complete* for a class of Kripke models $\mathcal{K}$, then $\neg\neg$**L** is strongly complete for the class of Kripke models that are *ultimately* in $\mathcal{K}$.

1 Introduction

Generally speaking, an *intermediate logic* is a logic that lies between intuitionistic logic and classical logic. For a language $\mathcal{L}$, an intermediate logic **L** includes intuitionistic logic and is closed under *modus ponens* and *generalization*. There is a large number of interesting intermediate logics that are studied in the logical literature, for example, [1], [2], [3], [4], [6], [7] and [9].

In this paper, we are interested in studying the logics that are extensions of the intuitionistic logic with doubly negated axioms. To be more precise, let **IPC**, **IQC**, **CPC** and **CQC** indicate propositional intuitionistic logic, first-order predicate intuitionistic logic, propositional classical logic, and first-order predicate classical logic respectively. The logics we are concerned with are $\mathbf{L} = \mathbf{IQC} + T$, where every sentence A in T is doubly negated, i.e., $A = \neg\neg B$, for some B in $\mathcal{L}$. By a well-known theorem of Glivenko, namely, the theorem that states that for any proposition A in a propositional language, $\mathbf{CPC} \vdash A$ if and only if $\mathbf{IPC} \vdash \neg\neg A$ (see, e.g., [8]), all the propositional intermediate logics with doubly negated axioms are exactly **IPC**. However, in first-order intermediate logics the situation is different.

A Glivenko-type theorem is proved for a logic, $\mathbf{QK} = \mathbf{IQC} + \neg\neg\forall x(A(x) \vee \neg A(x))$ in [9] and [6]. It states that for any first-order sentence A, $\mathbf{CQC} \vdash A$ if and only if $\mathbf{QK} \vdash \neg\neg A$. This results implies that any first-order intermediate logic **L** with doubly negated axioms is contained in **QK**, that is $\mathbf{L} \subseteq \mathbf{QK}$.

For a class of Kripke models $\mathcal{K}$, we associate another class of Kripke models $\mathcal{K}^u$, that are *ultimately* in $\mathcal{K}$. We prove that if a first-order intermediate logic **L** is strongly complete with respect to a class of Kripke models $\mathcal{K}$, then $\neg\neg$**L**, the logic axiomatised by the double negation of the universal closure of the axioms of **L**, is strongly complete with respect to the associated class of Kripke models $\mathcal{K}^u$. As corollaries (4.3 and 4.4) we deduce that:

- $\neg\neg\mathbf{DL} := \mathbf{IQC} + \neg\neg\forall\mathbf{x}((A \to B) \vee (B \to A))$ is sound and complete for Kripke models that are *ultimately linear*,
- $\neg\neg\mathbf{CD} := \mathbf{IQC} + \neg\neg\forall\mathbf{x}[\forall x(A \vee B(x)) \to (A \vee \forall x B(x))]$ is sound and complete for Kripke models that are *ultimately constant-domain*,
- $\neg\neg\mathbf{QJ} := \mathbf{IQC} + \neg\neg\forall\mathbf{x}(\neg A \vee \neg\neg A)$ is sound and complete for Kripke models that are *ultimately directed*,

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- **QK** $:=$ **IQC**$+\forall x\neg\neg A \to \neg\neg\forall x A$ (**IQC** + *Double Negation Shift*) is sound and complete for Kripke models that are *ultimately final*, i.e. Kripke models with the property that above each node, there exists a final node. This is a result first proved in [6].

- $\neg\neg$**KJC** $:=$ **QK** $+ \neg\neg$**CD** is sound and complete for Kripke models which are *ultimately final and constant domain*.

2 Preliminary definitions

We are concerned with countable first-order languages that include the connectives $\to, \vee, \wedge, \perp$ and the quantifiers $\exists, \forall$. The negation is defined as $\neg A := A \to \perp$. There are a countable number of function symbols, relation symbols, constant symbols and equality "$=$".

As usual, a sentence is a formula without free variables. For a first-order language $\mathcal{L}$ and a set of new constants C, $\mathcal{L}_C$ denotes the language $\mathcal{L}$ augmented with constants from C. For an arbitrary set of first-order formulae Γ and a first-order formula A, the relation "$\Gamma \vdash A$" means that there is a deduction of A from Γ in **IQC**$_{\mathcal{L}}$, where **IQC**$_{\mathcal{L}}$ is the intuitionistic first-order logic with equality that is formalized in the language $\mathcal{L}$, the language that contains all relations, functions and constant symbols appearing in $\Gamma \cup \{A\}$. Note that if $\mathcal{L}' \supseteq \mathcal{L}$ and $\Gamma \cup \{A\}$ is a set of sentences in $\mathcal{L}$, then $\Gamma \vdash_{IQC_{\mathcal{L}}} A$ iff $\Gamma \vdash_{IQC_{\mathcal{L}'}} A$. So we may use $\Gamma \vdash_{IQC} A$ without indicating the language $\mathcal{L}$ as subscript.

A first-order logic **L** in the language $\mathcal{L}$ is a set of formulae in $\mathcal{L}$ that is closed under the rules *modus ponens* and *generalization (universal quantification)*. All logics we are concerned with in this paper contain **IQC**. Hence such a logic can be indicated by $\mathbf{L} = \mathbf{IQC} + T$, for some set of sentences T in a language $\mathcal{L}$. For any set of formulae $\Gamma \cup \{A\}$ in $\mathcal{L}$, we may use the notation "$\Gamma \vdash_L A$" instead of "$\mathbf{L}, \Gamma \vdash A$". For any set of formulae Γ, we can assume a logic $\mathbf{L} = \bar{\Gamma}$, in which $\bar{\Gamma}$ is the closure of Γ under *modus ponens and generalization*. We call Γ the set of axioms for **L**.

To fix the notations, we recall [8] that a first-order Kripke model **K** for a language $\mathcal{L}$ (an $\mathcal{L}$-Kripke model), is a quintuple $\mathbf{K} = (K, \leq, D, \Vdash, \Phi)$, in which $(K, \leq)$ is a partially ordered set, D is a function on K that assigns an inhabited set $D(k)$ to each $k \in K$, and $\Vdash$ is a relation between elements of $k \in K$ and atomic sentences in the language $\mathcal{L}_k$, where $\mathcal{L}_k$ is an extension of the language $\mathcal{L}$ by constant symbols c_d, for every element d of $D(k)$, that is $\mathcal{L}_k = \mathcal{L} \cup \{c_d \mid d \in D(k)\}$. The class of mappings Φ is a collection of mappings $\{\varphi_{kk'} \mid k \leq k', k, k' \in K\}$, where $\varphi_{kk'} : D(k) \longrightarrow D(k')$, is such that φ_{kk} is the identity map and $\varphi_{k'k''} \circ \varphi_{kk'} = \varphi_{kk''}$. Moreover, if $k \leq k'$ and $k \Vdash A(d_1, \ldots, d_n)$, for an atomic formula A in the language $\mathcal{L}$ and $d_1, \ldots, d_n \in D(k)$, then $k' \Vdash A(\varphi_{kk'}(d_1), \ldots, \varphi_{kk'}(d_n))$. Then we can extend the relation $k \Vdash A$ to all formulae $A \in \mathcal{L}_k$. For more details on Kripke models, see [8].

3 Soundness and Completeness Theorems

Definition 3.1 For a first-order logic $\mathbf{L} = \mathbf{IQC} + T$, $\neg\neg\mathbf{L}$ is the logic $\mathbf{IQC} + \{\neg\neg A \mid A \in T\}$.

Now we define $\mathcal{K}^u$ for a class $\mathcal{K}$ of Kripke models. Informally speaking, $\mathcal{K}^u$ is the class of Kripke models that are ultimately in $\mathcal{K}$. More precisely:

Definition 3.2 Let $\mathcal{K}$ be a class of first-order Kripke models. $\mathcal{K}^u$ is the class of Kripke models $\mathbf{K} = (K, \leq, D, \Vdash, \Phi)$ such that for every node $k \in K$, there exists some $k' \in K$ with the following properties:

- $k \leq k'$ or $k > k'$,

- $\mathbf{K}_{k'} \in \mathcal{K}$, in which $\mathbf{K}_{k'}$ is the truncated model (restriction) of **K** with respect to k', i.e. the set of nodes of $\mathbf{K}_{k'}$ is $\{k'' \in K \mid k'' \geq k'\}$. All the other things for $\mathbf{K}_{k'}$ are inherited from **K**.

Theorem 3.3 *Let* $\mathbf{L} \supseteq \mathbf{IQC}$ *be a first-order logic in the language* $\mathcal{L}$ *and* $\mathcal{K}$ *be a class of* $\mathcal{L}$*-Kripke models such that* **L** *is sound in* $\mathcal{K}$*, i.e.* $\mathcal{K} \Vdash \mathbf{L}$*. Then* $\neg\neg\mathbf{L}$ *is sound in* $\mathcal{K}^u$*, i.e.* $\mathcal{K}^u \Vdash \neg\neg\mathbf{L}$*.*

Proof. It is an immediate consequence of the above Definitions. □

Recall that a logic $\mathbf{L} \supseteq \mathbf{IQC}_{\mathcal{L}}$ over the language $\mathcal{L}$ is said to be *complete* for a class of $\mathcal{L}$-Kripke models $\mathcal{K}$, if for every sentence A in $\mathcal{L}$ such that $\mathbf{L} \nvdash A$, there exists a Kripke model $\mathbf{K} \in \mathcal{K}$ such that $\mathbf{K} \Vdash \mathbf{L}$ and $\mathbf{K} \nVdash A$. And $\mathbf{L}$ is called to be *strongly complete* for $\mathcal{K}$, if for every set Γ of sentences and any sentence A (in the language $\mathcal{L}$), if $\Gamma, \mathbf{L} \nvdash A$, there exists a Kripke model $\mathbf{K} \in \mathcal{K}$ such that $\mathbf{K} \Vdash \Gamma, \mathbf{L}$ and $\mathbf{K} \nVdash A$.

Assume a first-order $\mathcal{L}$-Kripke model $\mathbf{K} = (K, \leq, D, \Vdash, \Phi)$ and let C be a set of new constants. We can extend $\mathbf{K}$ to an $\mathcal{L}_C$-Kripke model if for each $k \in K$, we can assign, to each $c \in C$, an element $\rho(k, c) \in D(k)$ such that it would be compatible with Φ. More precisely, we call a function $\rho : K \times C \longrightarrow \bigcup_{k \in K} D(k)$, *an assignment ρ of C in* $\mathbf{K}$, if for all $k, k' \in K$, $c \in C$, we have $\rho(k, c) \in D(k)$ and if $k \leq k'$, then $\varphi_{kk'}(\rho(k, c)) = \rho(k', c)$. For every countable set of new constants C and every assignment ρ of C in $\mathbf{K}$, we define a new Kripke model $\mathbf{K}^\rho = (K, \leq, D, \Vdash^\rho, \Phi)$ for the language $\mathcal{L}_C$, in which $\Vdash^\rho$ is defined as follows. For every node $k \in K$ and every atomic formula $A(x_1, \ldots, x_n)$ in the language $\mathcal{L}_{D(k)}$ and $c_1, \ldots, c_n \in C$,

$$k \Vdash^\rho A(c_1, \ldots, c_n) \text{ iff } k \Vdash A(\rho(k, c_1), \ldots, \rho(k, c_n)),$$

One can easily check that $\mathbf{K}^\rho$ fulfils the conditions of being a Kripke model and it is an extension of $\mathbf{K}$ to the language $\mathcal{L}_C$.

Let $\mathcal{K}$ be a class of $\mathcal{L}$-Kripke models and $\mathbf{L}$ be a logic in the language $\mathcal{L}_C$ (C is a fresh set of constants). We say that $\mathbf{L}$ is strongly complete for $\mathcal{K}$, iff for each set $\Gamma \cup \{A\}$ of $\mathcal{L}_C$ sentences, there exists some $\mathbf{K} \in \mathcal{K}$ and an assignment ρ of C in $\mathbf{K}$ such that $\mathbf{K}^\rho \Vdash^\rho \Gamma$ and $\mathbf{K}^\rho \nVdash^\rho A$.

Let $\mathbf{L}$ be a logic in the language $\mathcal{L}$ and C be a set of new constants. The logic $\mathbf{L}_C$ is the closure of $\mathbf{L} \cup \mathbf{IQC}_{\mathcal{L}_C}$ under modus ponens and generalization. Note that for a set of formulae $\Gamma \cup \{B\}$ in $\mathcal{L}_C$, we have $\mathbf{L}, \Gamma \vdash B$ iff $\mathbf{L}_C, \Gamma \vdash B$. First note that $\mathbf{L}_C$ is the set of formulae $A \in \mathcal{L}_C$ such that $\mathbf{L} \vdash A$, that is $\mathbf{L} \cup \mathbf{IQC}_{\mathcal{L}_C} \vdash A$. So $\mathbf{L}$ and $\mathbf{L}_C$ are equivalent logics over $\mathbf{IQC}_{\mathcal{L}_C}$. This means that for any $B \in \mathcal{L}_C$, $\mathbf{L} \vdash B$ iff $\mathbf{L}_C \vdash B$. This fact in combination with compactness theorem and deduction theorem implies that for any set of formulae $\Gamma \cup \{B\}$ in $\mathcal{L}_\mathcal{C}$ we have $\mathbf{L}, \Gamma \vdash B$ iff $\mathbf{L}_C, \Gamma \vdash B$.

Lemma 3.4 *Let $\mathcal{L}$ be a first-order language and C, E be two disjoint countably infinite set of new constants. Let $\mathbf{L} \supseteq \mathbf{IQC}$ be a logic in the language $\mathcal{L}$ such that $\mathbf{L}_C$ is strongly complete for a class of $\mathcal{L}$-Kripke models $\mathcal{K}$. Then for any set of sentences $\Gamma \cup \{A\}$ in $\mathcal{L}_{C \cup E}$, if $\Gamma, \mathbf{L}_C \nvdash A$, then there is a Kripke model $\mathbf{K} \in \mathcal{K}$ and an assignment ρ of $C \cup E$ in $\mathbf{K}$ such that $\mathbf{K}^\rho \Vdash^\rho \Gamma, \mathbf{L}$ and $\mathbf{K}^\rho \nVdash^\rho A$.*

Proof. Let $C = \{c_1, c_2, \ldots\}$. Assume the formula A. For each i, simultaneously replace all the occurrences of c_i in A with c_{2i} and let A_1 be the resulting formula. Do the same simultaneous substitutions for all the formulae in Γ and let Γ_1 be the resulting set of formulae. Then $\mathbf{L}, \Gamma_1 \nvdash A_1$, or equivalently $\mathbf{L}_C, \Gamma_1 \nvdash A_1$.

Let $E = \{e_1, e_2, \ldots\}$. Then in the formula A_1, for each i, simultaneously replace all the occurrences of e_i with c_{2i-1}. Let the resulting formula be A_2. Do the same simultaneous substitution for all the formulae in Γ_1, and let the resulting set of formulae be Γ_2. One can easily observe that $\mathbf{L} + \Gamma_2 \nvdash A_2$ or equivalently $\mathbf{L}_C + \Gamma_2 \nvdash A_2$. Then strongly completeness of $\mathbf{L}_C$ implies that there exists a Kripke model $\mathbf{K} := (K, \leq, D, \Vdash, \Phi)$ in $\mathcal{K}$ and an assignment ρ' of C in $\mathbf{K}$ such that $\mathbf{K}^{\rho'} \Vdash^{\rho'} \Gamma_2, \mathbf{L}_C$ and $\mathbf{K}^{\rho'} \nVdash^{\rho'} A_2$. Now we extract from ρ' an assignment ρ of $C \cup E$ in $\mathbf{K}$ with the desired properties. For each $i \in \mathbb{N}$ and $k \in K$, let $\rho(k, e_i) := \rho'(k, c_{2i-1})$. Also define $\rho(k, c_i) := \rho'(k, c_{2i})$. This finishes the definition of ρ. One can easily show that $\mathbf{K}^\rho \Vdash^\rho \mathbf{L}_C$. Also by definition of the assignments, for each formula $B(x_1, \ldots, x_n, y_1, \ldots, y_m) \in \mathcal{L}$, we have:

$$\mathbf{K}^\rho \Vdash^\rho B(c_1, \ldots, c_n, e_1, \ldots, e_m) \text{ iff } \mathbf{K}^{\rho'} \Vdash^{\rho'} B(c_2, \ldots, c_{2n}, c_1, \ldots, c_{2m-1}).$$

This implies that $\mathbf{K}^\rho \Vdash^\rho \Gamma$ and $\mathbf{K}^\rho \nVdash^\rho A$. □

Now we state and prove our main Theorem:

Theorem 3.5 *Let $\mathbf{L}' \supseteq \mathbf{IQC}$ be a first-order logic in $\mathcal{L}'$ and let C be a countably infinite set of fresh constants such that $\mathbf{L} := \mathbf{L}'_C$ is strongly complete for a class of $\mathcal{L}'$-Kripke models $\mathcal{K}$. Then $\neg\neg\mathbf{L}$ is strongly complete for $\mathcal{K}^u$.*

P r o o f. First define $\mathcal{L} := \mathcal{L}'_C$. Suppose $\Gamma \cup \{A\}$ is a set of sentences in $\mathcal{L}$ such that $\Gamma \nvdash_{\neg\neg\mathbf{L}} A$. Following the construction in ([8], pages 88-9) for **IQC**, we can construct the canonical Kripke model $\mathbf{K}^c := (K^c, \leq^c, D^c, \Vdash^c, \Phi^c)$ for $\neg\neg\mathbf{L}$, in which $\varphi^c_{kk'}$ is the identity map for all $k, k' \in K^c$ and for each $k \in K^c, C \subseteq D^c(k)$. As the language is countable, we may assume also that K^c and $D^c(k)$ are countable. By the above construction, we will also have $k \Vdash^c A$ iff $k \vdash_{\neg\neg\mathbf{L}} A$, for each $k \in K^c$. This implies that $\mathbf{K}^c \Vdash \Gamma$, $\mathbf{K}^c \Vdash \neg\neg\mathbf{L}$ and $\mathbf{K}^c \nVdash A$.

Informally speaking, for each $k \in K^c$ we do the following steps:

- we use the fact that $k \Vdash^c \neg\neg\mathbf{L}$ and find some sequence $k = k_1 \leq k_2 \leq \ldots \leq k_i$ of increasing nodes such that $k^* = \bigcup k_i \supseteq \mathbf{L}$.
- then for each formula B such that $k^* \nvdash B$, by the assumption of completeness, we find a Kripke model $\mathbf{K}_{k,B} \in \mathcal{K}$ such that $\mathbf{K}_{k,B} \Vdash k^*$ and $\mathbf{K}_{k,B} \nVdash B$. We then put $\mathbf{K}_{k,B}$ above all the k_i.

After these steps, we find an $\mathcal{L}'$-Kripke model $\mathbf{K}$. From this construction it is clear that $\mathbf{K} \in \mathcal{K}^u$. We also should provide an assignment ρ that tells how the new constants from C should be interpreted in $\mathbf{K}$. With this assignment, we finally show the desired result, i.e. $\mathbf{K}^\rho \Vdash \Gamma, \neg\neg\mathbf{L}$ and $\mathbf{K}^\rho \nVdash A$.

Now we give the details of the above steps. First fix some $k \in K^c$. By assumption of the language being countable, we may assume some enumeration $\{A_1, A_2, \ldots\}$ of the theorems of $\mathbf{L}$. We define inductively $k_i \in K^c$ such that $k_i \vdash_{\neg\neg L} A_1, \ldots, A_i$. Set $k_0 := k$, and suppose that we have already defined k_n. Since $k_n \vdash_{\neg\neg L} \neg\neg A_{n+1}$, we have $k_n \Vdash^c \neg\neg A_{n+1}$, and so there exists some $k_{n+1} \geq^c k_n$ such that $k_{n+1} \Vdash^c A_{n+1}$. This implies $k_{n+1} \vdash_{\neg\neg L} A_{n+1}$. Let $k^* := \bigcup_i k_i$ in the language $\mathcal{L}$ augmented with countable set of constants $D^*(k) := \bigcup_i D(k_i)$. Then by Lemma 3.4, for every $B \in \mathcal{L}_{D^*(k)}$ there exists some $\mathbf{K}_{k,B} = (K_{k,B}, \leq_{k,B}, D_{k,B}, \Vdash_{k,B}, \Phi^{k,B}) \in \mathcal{K}$ and some assignment $\rho_{k,B}$ of $D^*(k) \cup C$ in $\mathbf{K}_{k,B}$, such that $\mathbf{K}^{\rho_{k,B}}_{k,B} \Vdash \mathbf{L}, k^*$ and $\mathbf{K}^{\rho_{k,B}}_{k,B} \nVdash B$. We repeat the above construction for every $k \in K^c$.

Now we define $\mathbf{K} := (K, \leq, D, \Vdash, \Phi)$. We may assume that

- for every $k \in K^c$ and $B \in \mathcal{L}_{D^*(k)}$, $K^c \cap K_{k,B} = \emptyset$,
- for every $k, k' \in K^c$, $B \in \mathcal{L}_{D^*(k)}$ and $B' \in \mathcal{L}_{D^*(k')}$, if $(k, B) \neq (k', B')$, then $K_{k,B} \cap K_{k',B'} = \emptyset$.

Let

- $K := \bigcup_{k \in K^c, B \in \mathcal{L}_{D*(k)}} K_{k,B} \cup K^c$,
- $\leq := \left(\bigcup_{k \in K^c, B \in \mathcal{L}_{D*(k)}} \leq_{k,B} \cup \bigcup_{k \in K^c, B \in \mathcal{L}_{D*(k)}} (\{k\} \times K_{k,B}) \cup \leq^c\right)^{tc}$, where X^{tc} is the *transitive closure* of X. Note that with this definition if $k < k'$ and $k \in K^c$ and $k' \notin K^c$, then there exists some $l \in K^c$ and $B \in \mathcal{L}_{D*(l)}$ such that $k \leq^c l$ and $k' \in K_{l,B}$.
- $D := \bigcup_{k \in K^c, B \in \mathcal{L}_{D*(k)}} D_{k,B} \cup D^c$,
- $\Vdash := \bigcup_{k \in K^c, B \in \mathcal{L}_{D*(k)}} \Vdash_{k,B} \cup \Vdash^c$.

To finish the definition of $\mathbf{K}$, we need to define $\varphi_{kk'}$, and this will be done in the following way. If both k, k' are in K^c or some $K_{l,B}$, then we define $\varphi_{kk'}$ to be $\varphi^c_{kk'}$ or $\varphi^{l,B}_{kk'}$ respectively. Otherwise, there exists some $l \in K^c$ with $k \leq^c l$ such that $k' \in K_{l,B}$, and $k \leq k'$. Let $d \in D(k)$. It is clear that $\rho_{l,B}(k', d)$ is defined. So we set $\varphi_{kk'}(d) := \rho_{l,B}(k', d)$.

Now we prove the Theorem in the following items.

1. $\mathbf{K}$ is a Kripke model in $\mathcal{K}^u$.

   (a) Let $k, k', k'' \in K$ and $k \leq k' \leq k''$. We first show that $\varphi_{kk''} = \varphi_{k'k''} \circ \varphi_{kk'}$. We only treat the case where $k \in K^c$ and $k', k'' \notin K^c$. The other cases are treated similarly. Since $k' \notin K^c$, there exists some $l \in K^c$, $B \in \mathcal{L}_{D^*(l)}$ such that $k' \in K_{l,B}$, and so by definition of $\leq$, we have $k'' \in K_{l,B}$. Hence for each $d \in D(k)$, we have $\varphi_{kk''}(d) = \rho_{l,B}(k'', d)$ and $\varphi_{kk'}(d) = \rho_{l,B}(k', d)$. By using the properties of the assignment $\rho_{l,B}$ in the Kripke model $\mathbf{K}_{l,B}$, we get $\rho_{l,B}(k'', d) = \varphi^{l,B}_{k'k''}(\rho_{l,B}(k', d))$. Then by definition of $\varphi_{k'k''}$, we have $\rho_{l,B}(k'', d) = \varphi_{k'k''}(\rho_{l,B}(k', d))$, that implies $\varphi_{kk''}(d) = \varphi_{k'k''}(\varphi_{kk'}(d))$, as desired.

(b) For every atomic formula $B(x_1, \ldots, x_n)$ in the language $\mathcal{L}$ and $k, k' \in K$ and every $d_1, \ldots, d_n \in D(k)$,

$$k \leq k' \text{ and } k \Vdash B(d_1, \ldots, d_n) \text{ implies } k' \Vdash B(\varphi_{kk'}(d_1), \ldots, \varphi_{kk'}(d_n)).$$

The only non-trivial case is when $k \in K^c$ and $k' \notin K^c$, and we consider this case. If $k' \notin K^c$, then there exists some $l \in K^c$ and $B \in \mathcal{L}_{D^*(l)}$ such that $k' \in K_{l,B}$ and $k \leq^c l$. From $k \Vdash B(d_1, \ldots, d_n)$, we get $l \vdash_{\neg\neg L} B(d_1, \ldots, d_n)$, and by definition of l^*, we will have $l^* \vdash_{\neg\neg L} B(d_1, \ldots, d_n)$. Then $\mathbf{K}^{\rho_{l,B}}_{l,B} \Vdash B(d_1, \ldots, d_n)$. This implies $k' \Vdash_{l,B} B(\rho_{l,B}(k', d_1), \ldots, \rho_{l,B}(k', d_n))$, and hence $k' \Vdash B(\varphi_{kk'}(d_1), \ldots, \varphi_{kk'}(d_n))$, as desired. Thus $\mathbf{K}$ is a Kripke model.

(c) From the construction, it is clear that $\mathbf{K} \in \mathcal{K}^u$.

2. There exists some assignment ρ of C in $\mathbf{K}$ such that $\mathbf{K}^\rho \Vdash \Gamma$ and $\mathbf{K}^\rho \nVdash A$.

For each $k \in K^c$ and $c \in C$, define $\rho(k, c) := c$ and also for each $k \in K_{l,B}$, define $\rho(k, c) := \rho_{l,B}(k, c)$. It is enough to show that for each $k \in K^c$ and each formula $B(x_1, \ldots, x_n)$ in the language $\mathcal{L}'$ and constants $d_1, \ldots, d_n \in D(k)$,

$$k \Vdash B(d_1, \ldots, d_n) \text{ iff } k \Vdash^c B(d_1, \ldots, d_n).$$

We prove this by induction on the complexity of B.

If B is atomic, a conjunction, a disjunction or an existentially quantified formula, we can easily deduce the desired assertion. So let $B = B_1 \to B_2$. If $k \nVdash^c B(d_1, \ldots, d_n)$, then one can easily deduce by induction hypothesis that $k \nVdash B(d_1, \ldots, d_n)$. For the other way around, assume that $k \Vdash^c B(d_1, \ldots, d_n)$, or in other words, $k \vdash_{\neg\neg L} B(d_1, \ldots, d_n)$. We must show that for any $k' \in K$ such that $k \leq k'$, if $k' \Vdash B_1(\varphi_{kk'}(d_1), \ldots, \varphi_{kk'}(d_n))$, then $k' \Vdash B_2(\varphi_{kk'}(d_1), \ldots, \varphi_{kk'}(d_n))$. If $k' \in K^c$, then one can derive the desired by the induction hypothesis. So assume that $k' \notin K^c$. Then there exists some $l \in K^c$ and B, such that $k' \in K_{l,B}$ and $k \leq^c l$. Hence we have $l^* \vdash_{\neg\neg L} k$. This implies that $\mathbf{K}^{\rho_{l,B}}_{l,B} \Vdash_{l,B} B$, and hence $k' \Vdash B(\rho_{l,B}(k', d_1), \ldots, \rho_{l,B}(k', d_n))$. By definition of Φ, we then have $k' \Vdash B(\varphi_{kk'}(d_1), \ldots, \varphi_{kk'}(d_n))$. The case that B is universally quantified, is similar to the implication case.

□

4 Applications to some well-known intermediate logics

In this section, we will show how to apply Theorem 3.5 to some well-known first-order intermediate logics. Let us describe some of these well-known first-order intermediate logics [7], [5], [4], that we will consider in this paper.

- **DL** is defined to be **IQC** plus the universal quantification of the axiom schema $(A \to B) \vee (B \to A)$. This logic is known as *Dummett's Logic*.
- **CD** is defined to be **IQC** plus the universal quantification of the axiom schema $\forall x(A \vee B(x)) \to (A \vee \forall x B(x))$, in which x is not free in A. This logic is known as *Constant Domain Logic*.
- **QJ** is defined to be **IQC** plus the universal quantification of the *Weak Excluded Middle* axiom schema $\neg A \vee \neg\neg A$. This logic is known as *Jankov's Logic*.
- **QK** is defined to be **IQC** plus the universal quantification of the *Double Negation Shift* axiom schema $\forall x \neg\neg A(x) \to \neg\neg\forall x A(x)$. It is easy to verify that **QK** = **IQC** + $\neg\neg\forall x(A(x) \vee \neg A(x))$.
- **KJ** := **QJ** + **QK**.
- **KJC** := **QJ** + **QK** + **CD**.

Let us call a Kripke model $\mathbf{K} = (K, \leq, D, \Vdash, \Phi)$ to be *constant domain* iff for all $k, k' \in K$, and each $d' \in D(k')$, there exists some $d \in D(k)$ such that $\varphi_{kk'}(d) = d'$, in other words, $\varphi_{kk'}$ is surjective. This definition of constant domain Kripke models coincides with the standard definition (see, e.g., [8]) of this notion for Kripke models if $\varphi_{kk'}$ is assumed to be identity. Now one can easily observe that we have the following soundness theorems:

Theorem 4.1 ***(Soundness)*** *Let $\mathbf{K} = (K, \leq, D, \Vdash, \Phi)$ be an $\mathcal{L}$-Kripke model.*

- *If $(K, \leq)$ has a maximum element, then $\mathbf{K} \Vdash \mathbf{KJ}_{\mathcal{L}}$.*
- *If $(K, \leq)$ is linearly ordered, then $\mathbf{K} \Vdash \mathbf{DL}_{\mathcal{L}}$.*
- *If $(K, \leq)$ is directed, i.e., for all $k, k' \in K$, there exists some $k'' \in K$ such that $k, k' \leq k''$, then $\mathbf{K} \Vdash \mathbf{QJ}_{\mathcal{L}}$.*
- *If $(K, \leq)$ is ultimately final, i.e., for each $k \in K$, there exists some $k' \in K$ such that $k \leq k'$ and, for each $k'' \geq k'$, $k'' = k'$, then $\mathbf{K} \Vdash \mathbf{QK}_{\mathcal{L}}$.*
- *If $\mathbf{K}$ is constant domain, then $\mathbf{K} \Vdash \mathbf{CD}_{\mathcal{L}}$.*

Theorem 4.2 ***(Strong completeness)*** *For an arbitrary first-order language $\mathcal{L}$, $\mathbf{KJ}_{\mathcal{L}}$, $\mathbf{DL}_{\mathcal{L}}$, $\mathbf{QJ}_{\mathcal{L}}$, $\mathbf{CD}_{\mathcal{L}}$ and $\mathbf{KJC}_{\mathcal{L}}$ are strongly complete for $\mathcal{L}$-Kripke models with a maximum element, linear Kripke models, directed Kripke models, constant domain Kripke models and constant domain Kripke models with a maximum element, respectively.*

Proof. See [4], [2,7], [4], [1,3] and [4] for strong completeness of **KJ**, **DL**, **QJ**, **CD** and **KJC**, respectively. □

Let $\mathcal{K}_{KJ}$, $\mathcal{K}_{DL}$, $\mathcal{K}_{QJ}$, $\mathcal{K}_{CD}$ and $\mathcal{K}_{KJC}$, indicate the class of Kripke models with maximum element, linear Kripke models, directed Kripke models, constant domain Kripke models and constant domain Kripke models with maximum element, respectively. Then we can have the following result.

Corollary 4.3 *$\neg\neg$**KJ**, $\neg\neg$**DL**, $\neg\neg$**QJ**, $\neg\neg$**CD** and $\neg\neg$**KJC** are sound and strongly complete for $(\mathcal{K}_{KJ})^u$, $(\mathcal{K}_{DL})^u$, $(\mathcal{K}_{QJ})^u$, $(\mathcal{K}_{CD})^u$ and $(\mathcal{K}_{KJC})^u$, respectively.*

Proof. By Theorems 3.3, 3.5, 4.1 and 4.2. □

First-order classical logic, **CQC** is **IQC** plus the principle of excluded middle, i.e. $A \vee \neg A$. Let $\mathcal{K}_{CQC}$ be the class of Kripke models with a single node. Then **CQC** is strongly complete for $\mathcal{K}_{CQC}$. Hence by Corollary 4.3, $\neg\neg$**CQC** is strongly complete for $\mathcal{K}^u_{CQC}$. On the other hand, one can easily show that **QK** and $\neg\neg$**CQC** are equivalent over **IQC**. Hence

Corollary 4.4 **QK** *is strongly complete for ultimately final Kripke models.*

Acknowledgements We would like to thank G. Corsi for sending us her paper [2] and H. Ono for his very helpful comments on the first draft of this paper. We are very grateful of the anonymous referee for his or her very helpful comments, suggestions and corrections.

References

[1] M. Ardeshir, W. Ruitenburg and S. Salehi, Intuitionistic axiomatizations for bounded extension Kripke models, Annals Pure Appl. Logic **124**, 267-285 (2003).
[2] G. Corsi, Completeness Theorem for Dummett's LC Quantified and Some of Its Extensions, Studia Logica **51**, 317-335 (1992).
[3] S. Görnemann, A Logic Stronger Than Intuitionism, J. Symbolic Logic **36**, 249-261 (1971).
[4] G. Corsi and S. Ghilardi, Directed Frames, Arch. Math. Logic **29**, 53-67 (1989).
[5] D. M. Gabbay, Semantical Investigations in Heyting's Intuitionistic Logic (D. Reidel, Dordrecht, 1981).
[6] D. M. Gabbay, Applications of Trees to Intermediate Logics. J. Symbolic Logic **37**, 135-138 (1972).
[7] D. M. Gabbay, V. Shehtman, D. P. Skvortsov, Quantification in Nonclassical Logic, Studies in Logic and the Foundations of Mathematics, Volume 153, Elsevier, 2009.
[8] A. S. Troelstra and D. van Dalen, Constructivism in Mathematics (Vol.1, North-Holland, 1988).
[9] T. Umezawa, On Logics Intermediate between Intuitionistic and Classical Predicate Logic, J. Symbolic Logic **24**, 141-153 (1959).