In the name of God.

Homework-2 This set of exercises is due by 19/12/96.

Q1: Verify whether the following formulas are tautologies(always true sentences) or not. If not give a counter-example:

i) $\forall x\,(A(x) \vee B(x)) \quad \longleftrightarrow \quad \forall x A(x) \vee \forall x B(x)$

ii) $\exists x\,(A(x) \wedge B(x)) \quad \longleftrightarrow \quad \exists x A(x) \wedge \exists x B(x)$

iii) $\forall x\,(A \vee B(x)) \quad \longleftrightarrow \quad A \vee \forall x B(x)$

iv) $\forall x\,(A(x) \rightarrow B) \quad \longleftrightarrow \quad \exists x A(x) \rightarrow B$

v) $\forall x\,(A \rightarrow B(x)) \quad \longleftrightarrow \quad A \rightarrow \forall x B(x)$

Q2: If we define the limit of a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ on a point $a \in D_f$ as follows:

$\lim_{x\to a} f(x) = L \quad \longleftrightarrow \quad \forall \varepsilon > 0 \exists \delta > 0 : \mid x - a \mid < \delta \longrightarrow \mid f(x) - L \mid < \varepsilon$

Determine $\neg\,(\lim_{x\to a} f(x) = L)$ in terms of quantifiers, ε and δ and find a function $g : \mathbb{R} \longrightarrow \mathbb{R}$ such that it satisfies $\forall a \in \mathbb{R} \neg\,(\lim_{x\to a} g(x) = 0)$

Q3: Prove or disprove each of the following statements:

i) The set given by the application of the axiom of pairing on two arbitrary sets is unique.

ii) The set given by the application of the union axiom on an arbitrary set is unique.

iii) The set given by the application of the power-set axiom on an arbitrary set is unique.

iv) There exists a set A which its complement($B = \{x \mid x \notin A\}$) is a set.

v) For every pair of sets A, B we have: $\mathbb{P}(A \cup B) = \mathbb{P}(A) \cup \mathbb{P}(B)$

vi) For every pair of sets A, B we have: $\cap A \cap \cap B = \cap(A \cap B)$

vii) $A = \{X : \emptyset \notin X\}$ is a set.

Q4: Prove that the weaker form of the union axiom can be derived from the stronger version and other axioms.

Q5: Prove that there is no set containing all singletons(sets with 1 element). Then prove the generalized version of this statement for all $n \in \mathbb{N}$. Namely, prove that there is no set containing all n-element sets.
Hint: define each n-element set to be the disjoint union of a family of n singletons.

Q6: (*Bonus Problem*)Let A be a set-theoretical structure such that:

i) $\emptyset \in A$

ii) $\forall x \, (x \in A \rightarrow x \cup \{x\} \in A)$

iii) $\forall B \, (B \subseteq A \rightarrow \cup B \in A)$

Is A a set or not? In either case prove your answer.